OPTIMAL OPERATION OF MULTIQUALITY WATER SUPPLY SYSTEMS-III: THE Q-C-H MODEL

DANI COHEN[a], URI SHAMIR[b,*] and GIDEON SINAI[c]

[a]*Command and Control Unit, Mekorot Water Co., Ltd.,
9 Lincoln St., Tel Aviv, Israel;*
[b]*Faculty of Civil Engineering, Technion, Haifa 32000, Israel;*
[c]*Faculty of Agricultural Engineering, Technion, Haifa 32000, Israel*

A new technique for optimal operation of multiquality water supply systems is proposed. The technique, which is known as a Q-C-H (flow-quality-head) model, combines previously developed flow-quality (Q-C) and flow-head (Q-H) models for optimal operation of water supply systems. The decision variables in the model are the operation of treatment plants, pumps and valves. The model minimizes the cost of water at sources, treatment, energy, and loss of agricultural yield when water quality is low. The model uses an iterative modified projected gradient method combined with the Complex method. As in the Q-C and Q-H models, the solution method is based on decomposition, disaggregation/aggregation approach, involving internal and external optimization. The decision variables of the external model are the flows in the loops of the network and the removal ratios at the treatment plants. The operation of the pumps and valves are the decision variables of the internal model. The method is demonstrated by application to an example problem.

Keywords: Water supply systems; water quality; optimal operation of water systems; network analysis; hydraulic analysis

INTRODUCTION

Optimal operation of multiquality networks has been discussed in the previous papers of this series [4, 5]. Several earlier attempts to develop

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*Corresponding author.

models for this problem were somewhat limited. Sinai *et al.* [17] suggested decomposition of the complete solute transport problem which include 3 variables (Q-flow, C-concentration of water quality parameters, and H-head) into two simpler problems: Q-C ("chemical") and Q-H (hydraulic). They also suggested transformation of the Q and C variables to a solute flow variable $u = QC$ and to solve for it. The result is a limited Linear Programming (LP) model with hydraulic constraints, which can be used in networks where the dilution junction equation contains only one outflow unknown [17]. Mehrez *et al.* [11] used the same transformation of Sinai *et al.* [17] and developed a model for the optimal operation of multiquality networks. Their model is more advanced than that of Sinai *et al.* [17], yet it is also quite limited. This model is unable to simulate networks where the flow direction can be changed during the solution procedure. The same limitation arises in Mehrez *et al.*'s model [11]. In addition, it does not include treatment plants, and several constraints on C and H are missing. Ostfeld and Shamir [13] developed a more advanced model for the same problem. They used some of the mathematical tools developed by Cohen [2] and formulated a problem which was solved by a general software package GAMS/MINOS. This optimization technique does not make use of the properties of the problem, and therefore the utilization of this method is again limited.

In the present [2–5] the optimization problem is tackled by decomposition of the Q-C-H problem into two subproblems Q-C and Q-H. The two submodels are combined in this paper into a general Q-C-H model, using the solution methods developed earlier as building blocks.

Parameters of Water Quality and Consumer Types

Water quality may be described by primary and dependent quality parameters. Primary parameters are defined as those which are independent of other parameters. Dependent parameters are defined by a relationship between independent parameters, for example, SAR which is defined as the ratio between the concentration of Na, Mg and Ca in the water.

Water is supplied to 3 types of consumers:

1. Agricultural consumers, who have a crop yield function which depends on water quality [4].
2. Domestic and industrial consumers who do not require water treatment and do not incur costs from variable water quality at the consumer's connection.
3. Consumers with limits on the concentration of one or more water quality parameters. Such consumers require that quality be within specified limits, and no cost or benefit function for quality is specified.

Mathematical Representation of the Network

The water supply system can be described as a graph, consisting of arcs connected at nodes. See Cohen et al. [4] for a topological description of the network, and Cohen et al. [5] for definition of paths and the path matrix.

Flow Distribution in the Network

The optimal flow distribution is determined as part of the solution in the following manner. An initial flow distribution, which satisfies water continuity at all nodes is specified. This flow distribution is modified by the solution process, in a way in which continuity is retained. This maintenance of continuity is achieved by considering the circular flows, q, in loops and pseudo-loops, as decision variables, since when these flows are modified the continuity at nodes is maintained. As a consequence of this definition, the water flow continuity equations can be omitted, thus reducing the size of the optimization model. The vector q is of order n_1, the number of loops, with the ith component of the vector being the flow in the positive direction of loop i. Since the number of loops is considerably smaller than the number of pipes, using the circular flows in the loops rather than the flow in the pipes as the decision variables in this manner results in an even smaller model.

The relationship between the flows in all pipes of the network, $\mathbf{q}_a$, are defined by:

$$\mathbf{q}_a = \mathbf{q}_a^0 + L_a^T \mathbf{q} \tag{1}$$

where $\mathbf{q}_a^0$ is the initial flow distribution, which satisfies the water flow continuity equations, and $(\cdot)^T$ denotes transpose. The discharges to be supplied from the sources are defined by:

$$\mathbf{q}_s = \hat{A}\mathbf{q}_a = \hat{A}[\mathbf{q}_a^0 + L_a^T \mathbf{q}] \tag{2}$$

where $\hat{A}$ is a submatrix of A, obtained from those rows of A which relate to source nodes. The discharges through the valves, boosters, treatment plants and pump stations are:

$$\mathbf{q}_k = B_k^T \mathbf{q}_a = B_k^T [\mathbf{q}_a^0 + L_a^T \mathbf{q}] \quad k = p, b, t, v \tag{3}$$

Hydraulics of the Network Components

See Cohen *et al.* [5] for the equations of pipes, valves and pumps. A special method is developed to describe pump stations, as follows:

The head-discharge-efficiency function (operating surface) of a pump station depends on the particular set of pumps (called "a configuration"), which are in operation and their series-parallel interconnections. If these configurations are the decision variables, then the result is an Integer Nonlinear Programming problem, which is difficult or impossible to solve for a system of practical size. Cohen *et al.* [5] have shown that a pump station can be used more efficiently if the configuration also includes a control valve and by-pass. They also describe an alternative formulation resulting from the inclusion of a valve and by-pass for the operating surface of a pump station. The head-discharge relationship is defined by a smooth continuous function over a domain bounded by an envelope curve, h_p'' (q_p) (see Eq. (9) below), which is a limit on the head that can be supplied by the pump station as a function of its discharge. Each head discharge point under or on this envelope is defined with its efficiency, the least cost configuration for providing it, and the head which the control valve in the configuration has to dissipate. For a given q the efficiency as a function of the

head is expressed as:

$$\eta = \sum_{i=1}^{n} \eta_i \frac{h}{h_i} [\exp\{\varpi(\zeta_i - 1)\} + \exp\{\varpi(\zeta_i - 0.5)\}] \qquad (4)$$

where $\varpi(\cdot)$ is a normalized function defined by:

$$\varpi\{x\} = k_p \frac{x}{\sqrt{x^2 + \varepsilon_0}} \qquad (5)$$

h_i is the head obtained by configuration i when the discharge is q, and ζ_i is defined by:

$$\zeta_i = \sum_{j=1}^{i} \frac{\exp\{\varpi(h - h_j)\}}{\exp\{\varpi(h - h_j)\} + \exp\{-\varpi(h - h_j)\}} \qquad (6)$$

Pump stations usually consist of different pumps. Consequently, the envelope curve includes several nonsmooth points, which are smoothed as follows. Assume a pump station with n_i nonsmooth points $q^i \{i = 1, 2, \ldots, n_i\}$; each interval between two such points being derived from a different envelope function; denote by $g^i(q)$ the function for the interval (q^i, q^{i+1}). The smoothed envelope function is given by:

$$h_p'(q) = \frac{\sum_{i=1}^{n_i} g_i(q) \exp(\varpi\{\xi_i\})}{\sum_{i=1}^{n_i} \exp(\varpi\{\xi_i\})} \qquad (7)$$

where:

$$\xi_i = (q - q^i)(q^{i+1} - q) \qquad (8)$$

The $(q = 0, h = 0)$ point also has to be included. Inclusion of this point is achieved by bringing the envelope curve down to this point for $q < 1$ (an arbitrary value, considered small enough). Thus, the smoothed envelope curve, including the operating condition at zero flow, is:

$$h_p''(q) = h_p'(q_p) \frac{\exp(\varpi\{q - 1\}) + q^\alpha \exp(-\varpi\{q - 1\})}{\exp(\varpi\{q - 1\}) + \exp(-\varpi\{q - 1\})} \qquad (9)$$

where α is a coefficient which determines how steeply the curve is brought down to zero.

Cohen *et al.* [5] have shown that the solutions derived on the basis of this transformation of the discrete operating surface into a continuous and smooth one are practical and effective.

FORMULATION OF THE HYDRAULIC SUBPROBLEM

For a given flow distribution in the network, the power at each booster is expressed as a quadratic function of the discharge, and heads are computed with the Hazen-Williams equation (5, Eq. (12)). The maximum head in each pump station, h_p'', is given by Eq. (9) for the known flow. The pumping head can vary in the range $[0, h_p'']$. Each valve causes a head loss, even when fully open, denoted by $(h_v^*)^0$ (5, Eq. (14)).

Thus, for a given flow distribution the operation of the boosters is known and the optimal operation of the pump station and control valves, which realize the given flow, is formulated in the following Q_0-H problem [5]. The decision variables of this Q_0-H problem are; $\mathbf{h}_p$ – pumping heads at the n_p pump stations, $\mathbf{h}_v'$ – the additional head losses to those of fully open valves in the n_v control valves. Thus, the head losses in valves are determined from:

$$\mathbf{h}_v = \mathbf{h}_v' + (\mathbf{h}_v^*)^0 \quad \text{where } \mathbf{h}_v' \geq \mathbf{0} \qquad (10)$$

The constraints consist of the continuity of energy (Kirchoff's second law) and the head limits at nodes. Since there may be no feasible solution for the specified flow distribution in order to assure a mathematical solution, a pair of artificial variables with opposite signs, which are penalized in the objective function by coefficients θ^+ and θ^-, respectively, are added to each constraint. Denote by $\mathbf{x}_L^+$ and $\mathbf{x}_L^-$ the artificial variables added to the Kirchoff's law constraints, and denote by $\mathbf{x}_p^+$ and $\mathbf{x}_p^-$, the artificial variables added to the head constraints. The Q_0-H optimization problem now becomes:

$$\min f_0 = t k_e \alpha_p \mathbf{1}_p^T Q_p H_p E^{-1} \mathbf{1}_p + \theta^+(\mathbf{x}_L^+ + \mathbf{x}_p^+) + \theta^-(\mathbf{x}_L^- + \mathbf{x}_p^-) \qquad (11)$$

where t is the time period, k_e is the cost per unit energy, $\mathbf{1}_p$ is a vector of order n_p whose elements are 1, α_p is a constant whose value depends

on the units used (for SI units $\alpha_p = 104$), Q_p is a square diagonal matrix whose components $\mathbf{q}_p$ are the discharges at the n_p pump stations. H_p is a square diagonal matrix whose components $\mathbf{h}_p$ are the pumping heads at the n_p pump stations. E is a square diagonal matrix whose components η_p are the efficiencies.

Continuity of Energy (Kirchoff's Second Law)

$$L_p\mathbf{h}_p - L_v\mathbf{h}'_v + \mathbf{x}^+_L + \mathbf{x}^-_L = \mathbf{b}_p + L_a\mathbf{h}^*_a + L_v(\mathbf{h}^*_v)^0 - L_b\mathbf{h}^*_b \qquad (12)$$

where $\mathbf{b}_p$ is the head difference between the end of the loops: $b_p = 0$ when the loop is closed, and for a pseudo-loop b_p equals the known head difference between the ends of the pseudo-loop.

Head Limits

The head at each consumption node must be between a lower and an upper limit. The head at each consumption node is expressed by the energy equation along the path between the consumer node and a reference node at which the head is known independent of the flows (*e.g.*, a reservoirs or well)

$$\mathbf{h}'_c - P_a\mathbf{h}^*_a + P_v(\mathbf{h}^*_v)^0 - P_b\mathbf{h}^*_b \leq P_p\mathbf{h}_p - P_v\mathbf{h}'_v + \mathbf{x}^+_p + \mathbf{x}^-_p$$
$$\leq \mathbf{h}''_c + P_a\mathbf{h}^*_a + P_v(\mathbf{h}^*_v)^0 - P_b\mathbf{h}^*_b \qquad (13)$$

where h'_c and h''_c are the differences between the head at the reference node and the lower and upper allowable heads at the consumer, respectively.

Pumping Head Limits

At each pump station the pumping head is restricted to be less than the maximum specified at the given flow.

$$H_p \leq \mathbf{h}''_p \qquad (14)$$

with $\mathbf{h}''_p$ given by Eq. (9).

Non-negativity

$$\mathbf{h}_p, \mathbf{h}_v', \mathbf{x}_L^+, \mathbf{x}_L^-, \mathbf{x}_p^+, \mathbf{x}_p^- \geq \mathbf{0} \tag{15}$$

This Q-H_0 problem has a nonlinear objective function and linear constraints. The nonlinearity in the objective function is caused by the dependency of the efficiency of a pump on the pump head $\mathbf{h}_p$. However, this nonlinearity is normally mild over a wide range of $\mathbf{h}_p$. For given η_p, all the multipliers of h_p in the first term of Eq. (11) are therefore combined into a constant ρ^T, resulting in a LP problem with the objective function:

$$\min f_0 = \rho^T \mathbf{h}_p + \theta^+ (\mathbf{x}_L^+ + \mathbf{x}_p^+) + \theta^- (\mathbf{x}_L^- + \mathbf{x}_p^-) \tag{16}$$

where $\rho^T = t k_e \alpha_p Q_p E^{-1} 1_p$.

The Q_0-H problem is solved by what is known as the Algorithm Q_0-H, described by Cohen et al. [5]. The convergence of the Q_0-H algorithm has not been proven mathematically. However the algorithm has been applied over a wide range of cases and found to always converge in at most 3 iterations (Cohen [2]). The solution of this Q_0-H problem gives the optimal pumping head, $\mathbf{h}_p^*$, and the optimal head loss in valves, $\mathbf{h}_v^*$, for the given flow distribution. The optimal opening ratio of each valve is obtained from Eq. (9) when h_v^* and the known discharge are substituted. The optimal operation of each pump station to achieve the optimal pumping head h_p^* is obtained from Algorithm OPP, described by Cohen et al. [5], when h_p^* and the given flow are used. The optimal operation of each pump station determined in this manner includes the pumps to be operated and their series-parallel interconnections, the head loss dissipated by the control valve and the by-pass flow.

MATHEMATICAL MODEL OF THE Q-C-H PROBLEM

The network flow problem which includes all three variables, i.e., flow (Q), concentration of water quality parameter (C) and head (H) distribution is what is referred to as the Q-C-H problem. This problem

was decomposed into two subproblems: "chemical" (Q-C) and "hydraulic" (Q-H) ones which were the subject of the two previous papers in this series [4, 5]. The complete model of the Q-C-H problem makes use of some elements of the Q-C and Q-H models. These two submodels are connected by the flow (Q) distribution in the network. Solving the two submodels independently does not yield the same optimum as that of the combined Q-C-H model since there are some interactions in the solute transport problem that should be described in a complete Q-C-H model. The following is an attempt to present a mathematical model for optimizing the steady state operation of multi-quality water supply network where all the three variables Q, C, H are considered.

Decision Variables

The primary decision variables of the Q-C-H problem are: q – the circular flows; r – the removal ratios in the treatment plants (n_3 subvectors of primary quality parameters, where r^m is the removal ratios of the primary quality parameter m at the treatment plants (see [2−4]). c – the values of the primary parameters in the system (n_3 subvectors of primary quality parameters, where c^m is the concentration of the quality parameter m at the nodes); c_d – the values of the dependent quality parameters in the system (n_4 subvectors of dependent quality parameters, where c_d^m is the concentration of the dependent quality parameter m at the nodes).

The secondary decision variables are: the opening ratios of valves, configurations to be operated within the pump stations, head losses in control valves, and the by-pass flows. Recall that when q is known, these variables can be defined uniquely and directly from the solution of the Q_0-H problem.

Objective Function

The objective is to minimize total cost of operation of the system and losses incurred by reduction in productivity caused by lower water quality over a specified time period t. The total cost is made of the following components.

Water Supply Cost

This is the cost of water supplied from the sources. In general, the specific cost (per unit volume) varies with discharge. Thus, the specific cost of water at sources is given by a vector of functions, denoted by $w_s(q_s)$, whose dimension is the number of sources. When the unit cost at a source is fixed, the value of the function for that source is a constant. The total supply cost from the sources, ϕ_s, for the entire period t, is:

$$\phi_s = t w_s(q_s)^T q_s \tag{17}$$

Pumping Cost of Boosters

The total pumping cost of boosters, ϕ_b, for the time period is:

$$\phi_b = t k_e \alpha_b N_b(q_b)^T 1_b \tag{18}$$

where k_e is the cost per unit energy, α_b is a constant which depends on the units used (for SI units $\alpha_b = 104$), $N_b(q_b)$ is the power-discharge functions, and 1_b is a vector of order n_b whose elements are all 1.

Pumping Cost of Pump Stations

As described above, for a given flow distribution, defined by the circular flow, q, the pumping cost at pump stations is obtained from the solution of the Q_0-H problem, which is denoted by $\phi_p(q)$.

Treatment Cost

The total treatment cost, ϕ_t, is [4, Eq. (6)]:

$$\phi_t = t w_t(r)^T q_t \tag{19}$$

Yield Reduction Cost

The total loss due to yield reduction, ϕ_y, is [4, Eq. (10)]:

$$\phi_y = y_0^T B_0 [1_y - y] \tag{20}$$

where $\mathbf{y}$ and $\mathbf{y_0}$ are vectors with components y_i and $(y_0)_i$, respectively, and B_0 is the income matrix associated with full productivity with diagonal element i equal to $(\mathbf{b_0})_i$.

Penalty Costs

Some consumers have limits on the concentration of water quality parameters they can accept or use. These limits on water quality are introduced into the objective function by a penalty function, such that the solution is directed to be within the specified limits. For quality parameter m in node i, c_{mi}, for which the limits are denoted by c'_{mi} and c''_{mi} the penalty function is defined by:

$$P_{mi} = -z_{mi} \exp \left\{ \varpi \left(z_{mi} \right) \right\} \tag{21}$$

where

$$z_{mi} = \left(c_{mi} - c''_{mi} \right)\left(c_{mi} - c'_{mi} \right) \tag{22}$$

and $\varpi(\cdot)$ is the "normalized function" as defined in Eq. (5). Cohen et al. [3, 4] showed that this function is convex, continuous and smooth and can be controlled to prevent numerical difficulties often caused by penalty functions.

The penalty cost with respect to the primary parameter m, ϕ_L^m, is:

$$\phi_L^m = \sum_{i \in N} P_{mi}(c_{mi}) = \sum_{i \in N} -z_{mi} \exp\{\varpi(z_{mi})\} \quad N = N_1 + N_2 + N_3 \tag{23}$$

where the summation is only used over the nodes at which the value of the quality parameter m is restricted between limits. Similarly, the upper and lower limits, c''_{dmi} and c'_{dmi}, which are imposed on the dependent parameter m in node i are introduced in the objective function by a penalty term defined by:

$$P_{mi} = -e_{mi} \exp\{\varpi(e_{mi})\} \tag{24}$$

where

$$e_{mi} = \left(c_{dmi} - c''_{dmi} \right)\left(c_{dmi} - c'_{dmi} \right) \tag{25}$$

The penalty cost with respect to the dependent parameter m, ϕ_D^m, is:

$$\phi_D^m = \sum_{i \in N} P_{mi}(c_{dmi}) = \sum_{i \in N} -e_{mi}\exp\{\varpi(e_{mi})\} \quad N = N_1 + N_2 + N_3 \quad (26)$$

again with the summation only occurring across those nodes where the value of the water quality parameter m is restricted.

Combining the components mentioned above yields the objective function, f:

$$\min f = t\mathbf{w}_s(\mathbf{q}_s)^T\mathbf{q}_s + t\mathbf{w}_t(r_t)^T\mathbf{q}_t + t k_e \alpha_b \mathbf{N}_b(\mathbf{q}_b)^T\mathbf{1}_b +$$
$$+ \phi_p(\mathbf{q}) + \mathbf{y}_0^T B_0[\mathbf{1}_y - \mathbf{y}] + \sum_{m=1}^{n_3} \phi_L^m + \sum_{m=1}^{n_4} \phi_D^m \quad (27)$$

Constraints

The operator $\phi_p(\mathbf{q})$, which defines the solution of the problem Q_0-H, includes the hydraulic constraints: Kirchoff's second law, limits on the heads at consumers', limits on the pumping head at pump stations and limits on the opening ratio at valves. The additional constraints of the problem Q-C-H consist of the mass conservation at nodes, the dilution conditions, the relations between the dependent and primary parameters, limits on the discharges in various components of the network, and limits on the removal ratios at the treatment plants.

Mass Conservation Law

This law is expressed by n_3 (number of primary quality parameters) sets of equations. Each set includes equations for all nodes, except the source nodes.

$$\sum_{j \in E_1} a_{ij} q_a^j c_{jm}^+ + \sum_{i \in E_2} a_{ij} q_a^j (1 - \bar{r}_{jm}) c_{jm}^+ - d_i c_{im} = 0 \quad \forall i \in N_2 \ \& \ \forall m \in M_1$$

$$(28)$$

where d_i is the consumption at node i, M_1 is the primary quality parameters group, a_{ij} is the element ij of the adjacency matrix A, c_{jm}^+ is the concentration of primary quality parameter m in pipe j, which is

defined by the smoothed dilution conditions [3,4]:

$$c_{jm}^+ = \frac{c_{km}\exp\{\varpi(a_{ij}q_a^j)\} + c_{im}\exp\{-\varpi(a_{ij}q_a^j)\}}{\exp\{\varpi(a_{ij}q_a^j)\} + \exp\{-\varpi(a_{ij}q_a^j)\}} \tag{29}$$

where k and i are the end nodes of pipe j. This formulation overcomes the difficulty encountered by Shah and Sinai [15,16] in specifying the dilution conditions, by allowing the flows in each pipe to reverse direction during the iterations of the solution process as the optimal flow distribution is identified. r_m is the removal ratios of the primary parameter m at pipes belonging to E_2, defined by:

$$\tilde{r}_m = B_t^T \tilde{r}_m \tag{30}$$

Quality Parameter Relation Function

According to the definition of the dependent parameter group, each dependent water quality parameter has a function which defines its relationship with primary parameters. An example is SAR, which depends on the concentrations of Ca, Na and Mg. These functions are incorporated into the constraints:

$$c_{\dim} = \chi_{im}\{c_{i1}, c_{i2}, \ldots\ldots, c_{in}\} \quad \forall i \in N_1 \ \& \ \forall m \in M_2 \tag{31}$$

where M_2 is the dependent parameter group, $\chi_{im}\{\cdot\}$ is the relation between dependent parameter m and the primary parameters at node i.

Limits on Discharges

Boosters

The discharges through the boosters are limited:

$$0 \leq q_b \leq q_b'' \tag{32}$$

where q_b = booster flow.

Valves

Limits on the discharge through the valve are imposed to prevent cavitation, and to prevent reversal of flow if the valve is a one-way

valve.

$$\mathbf{q}'_v \leq \mathbf{q}_v \leq \mathbf{q}''_v \qquad (33)$$

where $\mathbf{q}''_v$ = maximum flow through valves.

For a one-way valve $q'_v = 0$, and for a bi-directional valve $q'_v = -q''_v$.

Pump Stations

The maximum discharge in each pump-station, q''_p, is obtained from the envelope curve, defined in Eq. (9):

$$0 \leq \mathbf{q}_v \leq \mathbf{q}''_p \qquad (34)$$

Using Eq. (3) constraints (26–28) transform into:

$$\mathbf{q}'_k - B_k^T \mathbf{q}_a^0 \leq B_k^T L_a^T \mathbf{q} \leq \mathbf{q}''_k - B_k^T \mathbf{q}_a^0 \quad \text{with } k = b, v, p \qquad (35)$$

for boosters, one-way valves and pump stations $\mathbf{q}'_b = \mathbf{q}'_v = \mathbf{q}'_p = 0$.

Sources

The upper limit, $\mathbf{q}''_s$, is either the supply capability of the source or some other restriction. The lower limit, $\mathbf{q}'_s$, is to prevent reverse flow or to prevent overflow if reverse flow into a reservoir occurs. The constraints at all sources are:

$$\mathbf{q}'_s \leq \mathbf{q}_s \leq \mathbf{q}''_s \qquad (36)$$

Using Eq. (2) this translates into:

$$\mathbf{q}'_s - \hat{A}\mathbf{q}_a^0 \leq \hat{A} L_a^T \mathbf{q} \leq \mathbf{q}''_s - \hat{A}\mathbf{q}_a^0 \qquad (37)$$

Limits on Removal Ratios

$$\mathbf{r}' \leq \mathbf{r} \leq \mathbf{r}'' \qquad (38)$$

where $\mathbf{r}'$ and $\mathbf{r}''$ are lower and upper limits on the removal ratios at the treatment plants for each primary water quality parameter. According to the definition of the removal ratio $r'' \leq 1$.

OPTIMIZATION STRATEGY

For a given flow distribution in the network and given removal ratios in the treatment plants the water quality values throughout the network are defined by Eq. (22), which becomes linear. The values of the dependent parameters can then be obtained from Eq. (31). The decision variables can thus be divided into two groups: the first contains the control variables, u, which consist of the vectors q and r; the second group, denoted by x, is the distribution of water quality values throughout the network, and are called the state variables (or resultant variables). Using this definition of terms, the Q-C-H problem has the general form.

$$(\text{Problem } P_0) \quad \min f_0(x, u) \tag{39}$$
$$\text{subject to:}$$

$$g(x, u) = 0 \tag{40a}$$

$$h' \leq h(u) \leq h'' \tag{40b}$$

$f_0(x, u)$ is the objective function given by Eq. (27). $g(x, u)$ are the mass conservation equations for the primary water quality parameters, including the dilution conditions with respect to the distribution of the quality parameters at nodes (Eqs. (28) and (29)). $h(u)$ are the functions of the circular flows and of the removal ratios constrained between bounds h' and h''. Equations (35) and (37) are related to the circular flows (q), while the constraints denoted by Eq. (38) are related to the removal ratios (r).

The Problem P_0 can be transformed into the equivalent problem:

$$(\text{Problem } P_1) \quad \min \Psi(u) = f_0(x(u), u) \tag{41}$$

$$\text{subject to:} \quad h' \leq h(u) \leq h'' \tag{42}$$

In other words, $\Psi(u)$ is a function of u and of x, such that x is obtained by solving the equations $g(x, u) = 0$, when u is given.

The Problem P_1 has *linear constraints* and a nonlinear objective function. The complexity of the Problem P_1 is however considerably

lower than that of the original problem, and therefore it is preferable to solve it.

PRINCIPLES OF THE SOLUTION METHOD

Problem P_1 can be solved using the projected gradient method. The main steps of the method are: (1) computation $\Psi(u)$; (2) computation of $\nabla_u \Psi$; (3) computation of the projected gradient if at the current u there are active constraints; (4) computation of the modified direction of change; (5) updating u if optimality conditions are not satisfied; and (6) using the Complex Method (Box, 1969 cited in [9]) when the algorithm gets "stuck" in a non-smooth point which is not optimal.

Computation of $\Psi(u)$

For a given $u = (q, r)$ the flows in all network arcs are known, and the costs of supply, treatment and pumping at boosters can be computed from Eqs. (17), (18) and (19), respectively. The pumping cost at the pump stations is obtained by solving the problem Q_0-H, as described by Cohen *et al.* [3, 5].

The yield loss cost ϕ_y and the penalty costs ϕ_L^i and ϕ_D^i depend on the water quality distribution which is found by solving the mass conservation equations ($g(x, u) = 0$). When q and r are given these equations are linear. Furthermore, they can be decomposed into one set for each water quality parameter. That is, for each water quality parameter there is a separate linear system of equations, all with the same coefficient matrix but with different right hand vectors. As a result, when q and r are given, the equation system $g(x, u) = 0$ has the following form:

$$Kc^m = b^m \quad m = 1, 2, \ldots, n_3 \tag{43}$$

where c^m is the distribution of quality parameter m at nodes. K is the specific quality discharge matrix whose components are computed by:

$$K_{ij} = \begin{bmatrix} d_{ij}Q_{ij} & \text{if } d_{ij}\,Q_{ij} \geq 0 \text{ and } i \neq j \\ -\sum_k d_{ik}\,Q_{ik} + d_i & \text{if } d_{ij}Q_{ij} < 0 \end{bmatrix} \tag{44}$$

Q_{ij} is the flow from node i to j, d_{ij} is element ij of the matrix D, and $\mathbf{b}^m$ is a vector of the mass flow input of primary parameter m, whose components are computed by:

$$(b^m)_i = \sum_{s \in N_1} Q_{si} \, c_s^m \qquad (45)$$

in which Q_{si} is the discharge from source node s to node i, and c_s^m is the concentration of parameter m at source node s.

Since each system of equations has the same matrix, solution by LU decomposition is preferable. That is, the matrix K is first decomposed into a lower triangular matrix K_L and an upper triangular matrix K_u such that $K = K_L K_u$. The distribution of the primary parameter m is obtained by solving

$$K_L \mathbf{w}^m = \mathbf{b}^m \qquad (46)$$

for $\mathbf{w}^m$ and

$$K_u \mathbf{c}^m = \mathbf{w}^m \qquad (47)$$

for $\mathbf{c}^m$. Since the matrices K_L and K_u are triangular, $\mathbf{w}^m$ and $\mathbf{c}^m$ can be obtained by forward and back substitution, respectively. The distribution of the dependent quality parameters are computed by the functions defined in Eq. (31). They yield loss cost ϕ_y and the penalty costs ϕ_L^m and ϕ_D^m are computed from Eqs. (20), (23) and (26), respectively.

Computation of the Gradient

The gradient of $\Psi(\mathbf{u})$ with respect to the control variables $\mathbf{u}$, $\nabla_u \Psi$, is:

$$\nabla_u \Psi = \nabla_u f + [\nabla_u \mathbf{g}]^T \beta \qquad (48)$$

where $[\nabla_u \mathbf{g}]$ is the Jacobian of Eq. (40a) with respect to $\mathbf{u}$, and the vector β is computed from:

$$[\nabla_x \mathbf{g}]^T \beta = -\nabla_x f \qquad (49)$$

$\nabla_u f$ consists of two subvectors $\nabla_q f$ and $\nabla_r f : \nabla_u f = [\nabla_q f \nabla_r f]^T$ where:

$$\nabla_r f = t F_r B_t [q_a^0 + L_a^T \mathbf{q}] \qquad (50)$$

in which F_r is the Jacobian of $\mathbf{w}_t(r)$ with respect to $\mathbf{r}$. The matrix is square and diagonal in which element ii is:

$$(F_r)_{ii} = d\mathbf{w}_t^i/d\mathbf{r}^i \qquad (51)$$

$\nabla_q f$ is defined by:

$$\nabla_q f = tL_a\hat{A}^T \left[F_s\hat{A}\left(q_a^0 + L_a^T\mathbf{q}\right) + \mathbf{w}_s \right]$$
$$+ tL_aB_t^T\mathbf{w}_t + \alpha_b tk_eL_bF_b\mathbf{1}_b + \nabla_q\phi_p \qquad (52)$$

where F_s and F_b are the Jacobian matrices of $\mathbf{w}_s$ and $\mathbf{N}_b$ with respect to $\mathbf{q}_s$ and $\mathbf{q}_b$, respectively. Both are square diagonal, with components defined, respectively, by:

$$(F_s)_{ii} = d\mathbf{w}_s^i/d\mathbf{q}_s^i \qquad (53)$$

$$(F_b)_{ii} = d\mathbf{N}_b^i/d\mathbf{q}_b^i \qquad (54)$$

$\nabla_q\phi_p$ is the gradient of the operator $\phi_p(\mathbf{q})$ which yields the optimal solution of the problem Q_0-H. Cohen *et al.* [3, 5] have shown that

$$\nabla_q\phi_p = \left[L_aS_aL_a^T - L_bS_bL_b^T + L_vS_vL_v^T \right]\pi_L$$
$$+ \left[L_aS_aP_a^T - L_bS_bP_b^T + L_vS_vP_v^T \right]\pi_p$$
$$+ L_pS_e\pi_e + \alpha_pk_et\left[I_p - Q_pE_p^{-1}(S_{\eta 1} + S_eS_{\eta 2}) \right]E_p^{-1}\mathbf{h}_p \qquad (55)$$

where π_L, π_p and π_e are the dual variables associated with Eqs. (12), (13) and (14), respectively.

S_a, S_b, S_v and S_e are square diagonal matrices of order n_a, n_b, n_v and n_p, respectively. Their components are defined by:

$$(S_a)_{ii} = 1.852 \, k_a^i \left(q_a^i\right)^{0.852} \qquad (56)$$

$$(S_v)_{ii} = \alpha_v^i \, k_v^i \left(q_v^i\right)^{\alpha_v - 1} \qquad (57)$$

$$(S_b)_{ii} = d\mathbf{h}_b^i/d\mathbf{q}_p^i \qquad (58)$$

$$(S_e)_{ii} = \left(d\mathbf{h}^0\right)_i/d\mathbf{q}_p^i \qquad (59)$$

I_p is a unit matrix of order n_p, $S_{\eta 1}$ and $S_{\eta 2}$ are the Jacobian matrices of the pumping efficiency with respect to $\mathbf{q}_p$ and $\mathbf{h}_p$, respectively. Both are square diagonal matrices of order n_p with components:

$$(S_{\eta 1})_{ii} = d\eta_p^i / dq_p^i \tag{60}$$

$$(S_{\eta 2})_{ii} = d\eta_p^i / dh_p^i \tag{61}$$

In the second term in Eq. (48), $[\nabla_u \mathbf{g}]$ is computed as follows. $[\nabla_u \mathbf{g}]$ is the Jacobian matrix of Eqs. (28) with respect to $\mathbf{u}$. It consists of n_3 pairs of submatrices; each is of order n_n, one is a derivative with respect to $\mathbf{q}$ and the other with respect to $\mathbf{r}$, where submatrix m is related to the primary parameter m. The vector β is computed by solving Eq. (40a).

The matrix $[\nabla_x \mathbf{g}]$ is block diagonal, where the matrix block m, $[\nabla_x^m \mathbf{g}]$, is related to primary parameter m. Furthermore, the submatrix is the specific input mass flow matrix, K, and therefore the vector β can be computed in separate parts:

$$K^T \beta^m = -\nabla_x^m f \quad \forall m \in M_1 \tag{62}$$

where β^m is subvector of β related to primary parameter m. $\nabla_x^m f$ is derived from the gradient of the yield loss cost and penalties with respect to the primary parameter m:

$$
\begin{aligned}
\nabla_x^m f &= \nabla_x^m \phi_y + \nabla_x^m \phi_L^m + \nabla_x^m \left(\sum_{j=1}^{n_4} \phi_D^j \right) \\
&= F_y^m + F_z^m + \left(\sum_{j=1}^{n_4} F_D^j + F_L^j \right) F_c^m
\end{aligned}
\tag{63}
$$

where F_y is the Jacobian matrix of the relative yield function with respect to the primary parameter m among the consumer nodes. Cell im contains the derivative of the relative yield function at node i with respect to primary parameter m:

$$(F_y)_{im} = -b_0^i y_0^i \frac{d\mathbf{y}^i}{dc_i^m} \tag{64}$$

F_y^m is the column of the matrix F_y related to primary parameter m. The matrix F_D is related to the dependent parameters and is defined similar to the matrix F_y:

$$(F_D)_{im} = -b_0^i y_0^i \frac{d\mathbf{y}^i}{d\mathbf{c}_{di}^m} \tag{65}$$

F_c is the Jacobian matrix of the dependent parameters with respect to the primary parameters. The matrix consists of n_3 submatrices. Submatrix m is defined with respect to primary parameter m whose cell kl is:

$$(F_c)_{kl} = \frac{d\chi_{kl}}{d\mathbf{c}_k^m} \tag{66}$$

F_z is the Jacobian of the penalty function ϕ_L with respect to primary parameters. Cell im contains the derivative of ϕ_L with respect to primary parameter m:

$$(F_z)_{im} = \frac{dP_{im}}{dc_m^i}$$

$$= -\exp\left\{\varpi(z_{im})\right\}\left[1 - \frac{k_p z_{im}\varepsilon}{z_{im}^2 + \varepsilon}\right]\left[c_{im}'' + c_{im}' - 2c_{im}\right] \tag{67}$$

in which z_{im} is computed from Eq. (22) according to c_{im} which is obtained from Eq. (28) for the given $\mathbf{q}$ and $\mathbf{r}$.

F_L is the Jacobian of the penalty function ϕ_D with respect to the independent parameters, and is defined similar to the matrix F_z:

$$(F_L)_{im} = \frac{d\hat{P}_{im}}{dc_{djm}^i}$$

$$= -\exp\left\{\varpi(e_{im})\right\}\left[1 - \frac{k_p e_{im}\varepsilon}{e_{im}^2 + \varepsilon}\right]\left[c_{dim}'' + c_{dim}' - 2c_{dim}\right] \tag{68}$$

in which e_{im} is computed from Eq. (25).

Since K in Eq. (62) has been decomposed into K_L and K_u, the subvector β^m can be computed simply by solving

$$K_u^T \mathbf{w}^m = -\nabla_x^m f \tag{69}$$

for $\mathbf{w}^m$ and then β^m is computed from

$$K_L^T \beta^m = \mathbf{w}^m \tag{70}$$

The matrices K_u and K_L are triangular, and thus $\mathbf{w}^m$ and β^m can be obtained by back and forward substitution, respectively.

Computation of the Projected Gradient

Following Rosen [14], the projected gradient, $\mathbf{s}_u$, is:

$$\mathbf{s}_u = -\left[\nabla_u \Psi + N_a^T \lambda\right] \tag{71}$$

in which N_a is the coefficient matrix of the active constraints, and λ is obtained from

$$N_a^T N_a \lambda = -N_a^T \nabla_u \Psi \tag{72}$$

In the present problem N_a has the form

$$N_a = \frac{A_a \quad 0}{0 \quad J_e}$$

where A_a is the coefficient matrix of the active set related to $\mathbf{q}$, and J_e is the coefficient matrix of the active set related to $\mathbf{r}$. $\nabla_u \Psi = [\nabla_q \Psi; \nabla_r \Psi]$, thus the system described by Eq. (72) can be decomposed into two subsystems:

$$A_a A_a^T \lambda_1 = -A_a \nabla_q \Psi \tag{73}$$

$$J_e J_e^T \lambda_2 = -J_e \nabla_r \Psi \tag{74}$$

where λ_1 and λ_2 are sub-vectors of λ. Since $J_e J_e^T = I$ then the projected gradient with respect to $\mathbf{r}$, $\mathbf{s}_r$, is

$$\begin{array}{ll} \mathbf{s}_r^i = 0 & \text{if } i \in I_r \\ \mathbf{s}_r^i = -\nabla_r \Psi & \text{if } i \notin I_r \end{array} \tag{75}$$

where I_r is the active set with respect to $\mathbf{r}$.

There are several methods for computing $\mathbf{s}_q$, the projected gradient with respect to $\mathbf{q}$. Cohen, et al. [3, 4] show that the projected gradient

can be computed directly as follows. First, the matrix $A_a A_a^T$ is decomposed into A_L and A_U such that $A_a A_a^T = A_L A_U$, and then λ_1 is computed by solving

$$A_L \mathbf{w} = -A_a \nabla_q \Psi \tag{76}$$

for $\mathbf{w}$ and λ_1 is obtained from

$$A_U \lambda_1 = \mathbf{w} \tag{77}$$

by back and forward substitution, respectively. Substituting λ_1 into Eq. (71) yields

$$\mathbf{s}_q = -\left[\nabla_q \Psi + A_a^T \lambda_1 \right] \tag{78}$$

As long as the active set does not change, no recomputation of A_L and A_U is needed, and the vector λ_1 is computed for each new $\nabla_q \Psi$. The matrices of A_L and A_U are updated if a constraint is added to the active set, and a new decomposition is required only when the active set is reduced, which occurs only when the current $\|\mathbf{s}_q\|$ equals zero. This aspect of the algorithm reduces the computational burden of the model significantly.

Computation of the Modified Projected Gradient

The removal ratios at the treatment plants are in the range $[0, 1]$, whereas the circular flows can be of the order of tens, hundreds and even thousands. Furthermore, the effect of a unit change in a removal ratio is much greater than the effect of a unit change in the circular flow. As a result, there is a scaling problem between $\mathbf{r}$ and $\mathbf{q}$. Methods for nonlinear optimization are in general sensitive to scaling, and especially so the projected gradient which is an extension of steepest decent. Gill *et al.* [8] (pp. 273–275) have suggested transformation of the variables as a method to overcome these difficulties. In this problem transformation can be performed by defining $\mathbf{r}$ in percent (%) when the circular flows are in the order of hundreds or in parts per thousand (‰) when the circular flows are in the order of thousands. However, attempts to apply this approach to the problem failed, and the computational difficulties remained. The alternative approach used in the paper is to note that, for a separable quadratic problem, the

optimal solution is achieved in one iteration, independent of the initial point and the order of the scaling. As shown above, s_r and s_q are orthogonal. Therefore the problem is both quadratic and separable with respect to q and r, and the search direction can be found by the following two-phase algorithm.

Algorithm TF (Two Phases)

Step a   Assume initial search directions $\mathbf{d}'_q = \mathbf{s}_q$ for q, and $\mathbf{d}'_r = \mathbf{s}_r$ for r.

Step b   Compute α''_q, the maximal step along $\mathbf{d}'_q$, and α''_r, the maximal step along $\mathbf{d}'_r$, such that $\left(\mathbf{q} + \alpha''_q \mathbf{d}'_q\right)$ and $\left(\mathbf{r} + \alpha''_r \mathbf{d}'_r\right)$ do not violate a constraint which is not included in the active set.

Step c   Compute α^*_q, the optimal step length along $\mathbf{d}'_q$:

$$\Psi\left(\mathbf{q} + \alpha^*_q \mathbf{d}'_q, \mathbf{r}\right) = \min\left\{\Psi(\mathbf{q} + \alpha_q \mathbf{d}'_q, \mathbf{r}) \big| 0 \le \alpha_q \le \alpha''_q\right\}$$

The computation of the optimal step length is discussed below.

Step d   Compute α^*_r, the optimal step length along $\mathbf{d}'_r$:

$$\Psi\left(\mathbf{q}, \mathbf{r} + \alpha^*_r \mathbf{d}'_r\right) = \min\left\{\Psi(\mathbf{q}, \mathbf{r} + \alpha_r \mathbf{d}'_r) \big| 0 \le \alpha_r \le \alpha''_r\right\}$$

Note that the one dimensional search along q and r independently and separately remains on the manifold of the active set.

Step e   Compute the norm of the updated direction, s:

$$\|\mathbf{s}\| = \left[\alpha^*_q (\mathbf{d}'_q)^T \mathbf{d}'_q + \alpha^*_r (\mathbf{d}'_r)^T \mathbf{d}'_r\right]^{1/2}$$

Step f   Compute $\mathbf{d}_q$, the normalized modified projected gradient direction related to q:

$$\mathbf{s}_q = \alpha^*_q \mathbf{d}'_q / \|\mathbf{s}\|$$

Step g   Compute $\mathbf{d}_r$, the normalized modified projected gradient direction related to r:

$$\mathbf{s}_r = \alpha^*_r \mathbf{d}'_r / \|\mathbf{s}\|$$

Note that $\mathbf{d}_r = \mathbf{d}'_r$ if $\mathbf{s}_q = 0$ and $\mathbf{d}_q = \mathbf{d}'_q$ if $\mathbf{s}_r = 0$.

The optimal step length in steps c and d could be determined by
one of the methods for one dimensional search, such as the Powell or
Davidon methods (Avriel [1], pp. 221 – 240). When the optimality con-
ditions are not met, a one dimensional search is performed along the
updated direction, s, such that an optimal step length is obtained. In
some cases, the computation of Ψ and $\nabla\Psi$ can be computationally
intensive and therefore a non-exact search method is preferable. The
efficiency of the scaling method is probably reduced for non-exact
methods, since its efficiency for separable quadratic problem is derived
for an exact one dimensional search. This was examined over a wide
range of examples, and it was found that the overall efficiency of the
algorithm is good.

Incorporation of the Complex Method

So far it has been shown that the Q-C-H problem can be formulated as
a optimization problem in the subspace of q and r with a nonlinear
objective function and linear constraints. For a given q and r the opti-
mal operation of the pump stations and of the control valves are de-
termined by solving the Q_0-H problem. The gradient of the objective
function is computed analytically as detailed above. Eiger et al. [6]
showed that the objective function in such an "inner-outer" optimiza-
tion problem may have non-smooth points. In the present problem
this is due to the inclusion of $\phi_p(q)$, the solution of the problem Q_0-H.
The objective function of the outer optimization is neither convex nor
concave, with linear constraints. Thus, even the application of a non-
smooth optimization method can not guarantee a solution.

The Complex Method was developed by Nelder and Mead [12]
for unconstrained problems. Box (1969), cited in Jacoby et al. [9], ex-
tended it to problems with inequality constraints. The method is based
on the objective function values and does not use gradients, so it is ap-
plicable to non-smooth problems. The efficiency of the method is im-
proved when the feasible region is convex, since the search process
is guaranteed to remain in the feasible domain. This is the case in the
present problem since the constraints are linear. The efficiency of the
Complex method is reduced substantiay, however, as the number of
decision variables increases. The modified projected gradient and the

Complex methods are therefore combined. The process is performed by the modified projected gradient while the projected gradient is effective, *i.e.*, while it provides the descent direction. When the process approaches a non-smooth (kink) point the gradient may show incorrectly the descent direction, since the gradient is only one of the members of the subdifferential set. In this situation the one dimensional process terminates with a null step where the Kuhn-Tucker conditions are not satisfied. At this point the Complex method is used to reach a better point, with a search based only on function values.

Termination Criteria

When the relative change of the objective function at two successive points is less than a set small value ε_c or/and relative change in the decision variables in two successive iterations is less than another set small parameter ε_v, then the Kuhn-Tucker optimality condition is satisfied if the multipliers with respect to the active constraints at their upper bounds are nonnegative, and non-positive for the active constraints at their lower bounds. Otherwise, the active constraint having the multiplier with maximal absolute value is dropped from the set, and the solution process continues with respect to the updated active set. However, it is possible that the current point is non-smooth and then the Kuhn-Tucker conditions are not true for the optimal point. Thus if the subdifferential set consists of a null projected subgradient the current point is optimal, otherwise the Complex method is used to reach a better point, with a search based only on function values. If no improved point is found, then the current point is the best that can be found (from the initial point).

Algorithm Q-C-H

Initialize Assume initial value $u^0 := \{q^0, r^0\}$. Initialize the iteration counter $k := 0$.

Step 1 Compute the distribution of the primary quality parameters by solving:

$$Kc^m = b^m$$

Step 2   Compute the distribution of the dependent quality parameter by using the relation functions:

$$c_{dnm} = \chi_{nm}(c_{n1}, c_{n2}, \ldots, c_{nm}) \quad \forall n \in N_3 \quad \text{and} \quad \forall m \in M_2$$

Step 3   Solve the Q_0-H problem at the current flow distribution $\mathbf{q}^k$.
Step 4   Compute the value of the objective function, $\Psi(\mathbf{u}^k)$.
Step 5   Compute $\nabla_x f$ from Eq. (63).
Step 6   Compute β^i by solving:

$$K^T \beta^m = -\nabla_x^m f \quad \forall m \in M_1$$

Step 7   Compute $\nabla_q f$ and $\nabla_r f$ from Eqs. (52) and (50), respectively.
Step 8   Compute $\nabla_q \Psi$ and $\nabla_r \Psi$ from:

$$\nabla_q \Psi = \nabla_q f + [\nabla_q \mathbf{g}]^T \beta$$

$$\nabla_r \Psi = \nabla_q f + [\nabla_r \mathbf{g}]^T \beta$$

Step 9   Compute λ_1 by solving:

$$A_L A_U \lambda_1 = -A_a \nabla_q \Psi$$

Step 10   Compute λ_2 from $\lambda_2 = -\nabla_r \Psi$
Step 11   Compute the projected gradient with respect to $\mathbf{q}$ from:

$$\mathbf{s}_q = -\left[\nabla_q \Psi + A_a^T \lambda_1\right]$$

and with respect to **r** from:

$$\mathbf{s}_r^i = 0 \qquad \text{if } i \in I_r$$
$$\mathbf{s}_r^i = -\nabla_r \Psi \quad \text{if } i \notin I_r$$

Step 12   Compute the modified projected gradient as described in Algorithm TF.
Step 13   Compute the optimal step length along the modified projected gradient, α^*.
Step 14   Check the optimality conditions. If satisfied goto Step 16, otherwise goto Step 15.

Step 15   Updating: If the active set has changed, update the active matrix A_a and the decomposition matrices A_L and A_U. Update $\mathbf{q}$ and $\mathbf{r}$ from $\mathbf{q}^{k+1} := \mathbf{q}^k + \alpha^* \mathbf{d}_q$ and $\mathbf{r}^{k+1} := \mathbf{r}^k + \alpha^* \mathbf{d}_r$, update the iteration counter: $k := k+1$, and return to Step 1.

Step 16   Complex search: Use the Complex procedure. If an improved point is found, go back to Step 0 with the improved point as the initial flow distribution, otherwise, go to Step 17.

Step 17   Compute the optimal operation of the valves with $\mathbf{q}_v$ and $\mathbf{h}_v$ at the current point substituted. $\mathbf{q}_v$ is defined from Eq. (3) according to $\mathbf{q}^k$, and $\mathbf{h}_v$ is obtained from the solution of the problem Q-H$_0$ at the current point.

Step 18   Compute at each pump station the optimal configuration, the optimal head loss dissipated by the pumping control valve and the optimal by-pass flow by using the algorithm OPP, described by Cohen et al. [3, 5] where $\mathbf{q}_p$ and $\mathbf{h}_p$ are considered. Note that $\mathbf{q}_p$ is defined from Eq. (3) according to $\mathbf{q}^k$, and $\mathbf{h}_p$ is obtained from the solution of the problem Q$_0$-H at the current point.

EXAMPLE

The technique described in the previous sections was applied to the network shown schematically in Figure 1. The network consists of 9 pipes, 9 nodes, 2 pump stations, 2 boosters, and 4 control valves and is fed from two constant head reservoirs at nodes 8 and 9, and delivers to consumers at nodes 4, 5, 6 and 7. The network is operated 2000 hours and the cost per unit energy is 0.22 NIS/kWh ($1 US = 3.6 NIS, NIS = New Israeli Shekel).

The water quality parameters of importance in this system are salinity, magnesium and sulphur. The relevant quantity and quantity data for the sources are given in Table I.

At the consumption nodes 4, 5 and 6 a quadratic yield function of the following form was defined with respect to salinity $y = a_0 + a_1 c + a_2 c^2$, in which y is the relative yield, c is the salinity in mg-cl/l, and a_0, a_1 and a_2 are constants as shown in Table II. At node 7 salinity is
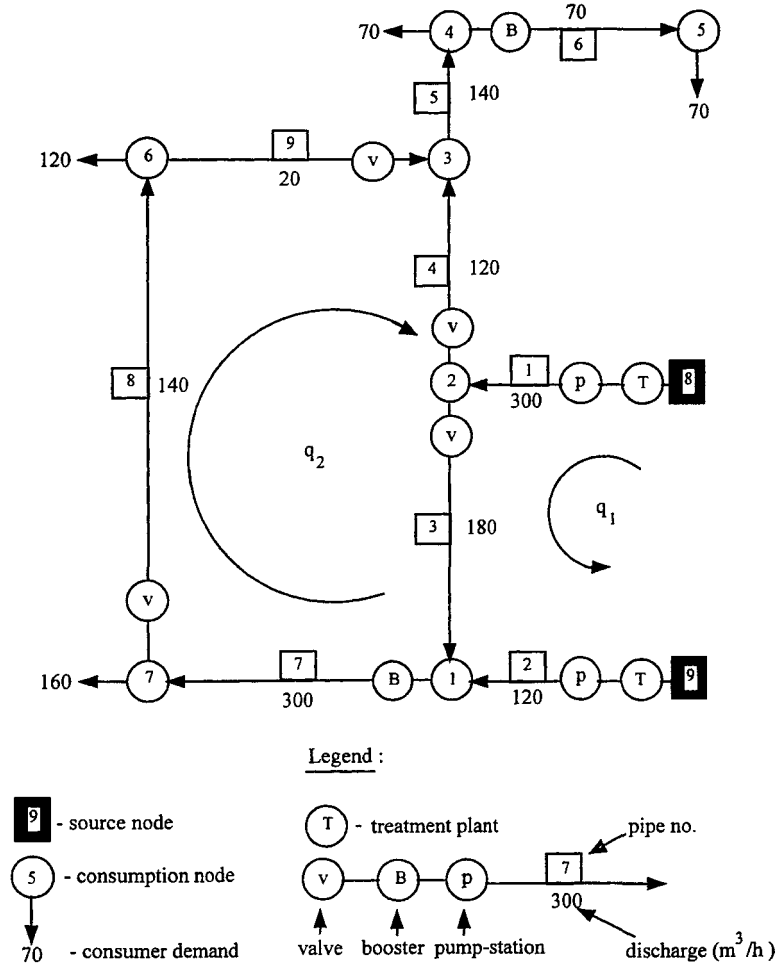
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FIGURE 1   Schematic description of the network.

TABLE I   Data for sources

| Node | Specific cost NIS/m³ | Elevation* m | Maximal discharge m³/h | Water quality | | |
				Salinity mgcl/l	Magnesium mg/l	Sulphur mg/l
8	0.638	− 252.5	325	450	140	500
9	0.256	− 255.0	700	860	250	300

* Elevations are negative because the system is in the Jordan Valley, below sea level.

TABLE II   Data for consumption nodes with respect to quality

| Node | $b_0 * y_0$ | Coefficients of the yield function | | | Upper limits for water quality | | |
| | | | | | Salinity | Magnesium | Sulphur |
–	$NIS * 10^{-8}$	a^0	$a_1 * 10^5$	$a_2 * 10^8$	$mgcl/l$	mg/l	mg/l
4	0.700	1.0	1.250	– 2.707	–	250	500
5	0.448	1.0	– 3.06	– 3.890	–	200	500
6	1.200	1.0	1.250	– 2.707	–	200	450
7	–	–	–	–	600	170	450

Note: $b_0 * y_0$ is the income when there is no yield loss due to water quality.

TABLE III   Hydraulic data for consumption nodes

Node	Consumption m^3/h	Elevation m	Minimum pressure head m	Maximum pressure head m
4	70	– 258	40	80
5	70	– 217	40	80
6	120	– 242	35	80
7	160	– 246	40	80

constrained by an upper bound. Upper limits as shown in Table II were defined for the other water quality parameters.

Other relevant data for the consumption nodes are given in Table III.

Treatment plants for salinity and magnesium are placed on pipe 2, and a treatment plant for sulphur is placed on pipe 1. The cost with respect to treatment (removal) of salinity is $w_t = 2.151 * 10^{-4} r^2$ and with respect to magnesium and sulphur is $w_t = 1.32 * 10^{-4} r^2$ in which w_t is the specific cost in NIS/m^3, and r is the removal ratio in %. All the removal ratios are limited to 75%.

Identical control values with the following head loss function were placed on pipes 3, 4, 9 and 8. $h_v = 10^{-6} q_v^2 m_v^{-1.5}$, in which h_v is the head loss in m, q_v is the discharge in m^3/h and m_v is the opening ratio. Boosters, each with cubic head-discharge function, are placed on pipes 6 and 7. $h_b = a_0 + a_1 q_b + a_2 q_b^2 + a_3 q_b^3$ and linear power-discharge function $N_b = b_0 + b_1 q_b$ where h_b is the head in m, N_b is the power in kW, and q_b is the discharge in m^3/h. The values of the coefficients $a_i (i = 0, \ldots, 3)$ and $b_i (i = 0, 1)$ for the head discharge and power discharge functions are given in Table IV.

The data for pipes are given in Table V.

All pipes have Chw = 120.

TABLE IV   Coefficients of the head-discharge and power-discharge functions of boosters

Booster	Placed on pipe	a_0	a_1	$A_2 * 10^4$	$A_3 * 10^7$	B_0	B_1
1	6	34.765	−0.025	2.3175	−16.26	16.303	0.0109
2	7	17.382	−0.012	1.1587	−0.813	18.795	0.0054

TABLE V   Data for pipes

Pipe	1	2	3	4	5	6	7	8	9
length (m)	400	1300	3700	1000	4300	2600	800	5000	3500
diameter (mm)	250	250	300	250	300	250	250	250	250
initial flow (m^3/h)	300	120	180	120	140	70	300	140	20

The network consists of 2 loops: one closed loop of pipes 3, 7, 8, and 9, and one pseudo-loop between the nodes 8 and 9 which includes pipes 1, 2, and 3. The initial flows, the positive directions of the pipe flows and of the circular flows are shown in Figure 1.

The pump station placed on pipe 1 (pump station A), consists of 3 pumps of type a and one pump of type b. The pump station placed on pipe 2 (pump station B), consists of 2 pumps of type c and one pump of type d. The head-discharge and efficiency-discharge functions for the four types of pipes are given in Figures 2 and 3. Only parallel connections between the pumps are possible.

The optimal solution is $f^* = 647108.83$ NIS, and the optimal circular flows are $q_1^* = -88.19$, $q_2^* = -14.09\,m^3/h$ (flows relative to those in Fig. 1). The optimal distributions of flows and quality parameters are given in Figure 4.

The optimal removal ratios of salinity and magnesium in the treatment plants placed on pipe 2 are 23.72% and 25.72%, respectively. The optimal removal ratio of sulphur in the treatment plant placed on pipe 1 is zero. The optimal operation of the pump station and the pressure heads at the consumer nodes are summarized in Tables VI and VII, respectively.

The solution was obtained after 27 iterations in 4.39 sec on an IBM-3081 computer. The values of the objective function during the solution process are shown in Figure 5. The results are presented in Tables VI–IX.
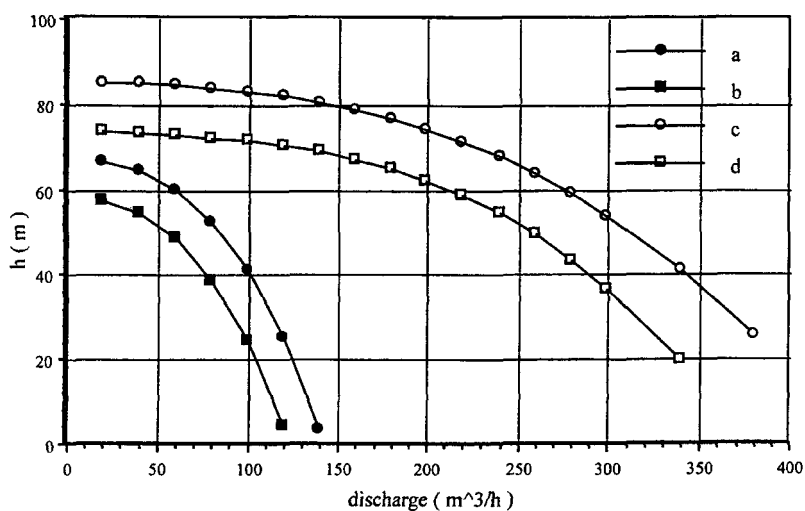
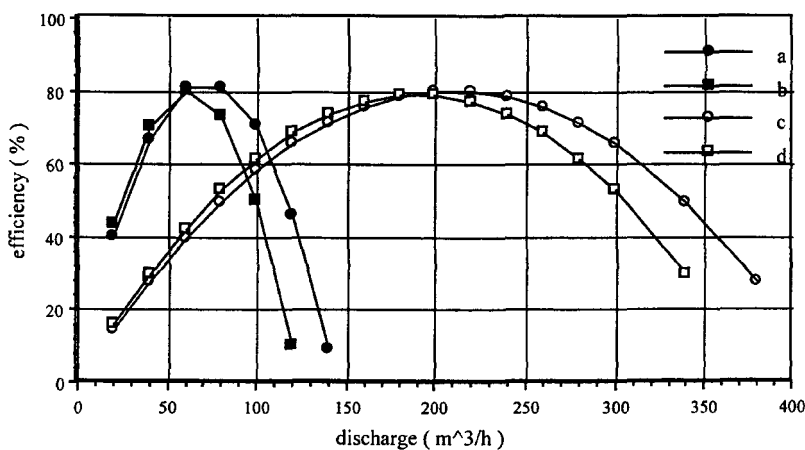FIGURE 2   Head-discharge functions for the pumps in the pump stations.

FIGURE 3   Efficiency-discharge functions for the pumps in the pump stations.

The method was also applied to the water distribution system of Central Arava Region, which supplies an area in the southern region of Israel. This system has 38 nodes, 39 pipes, 11 sources (wells), 9
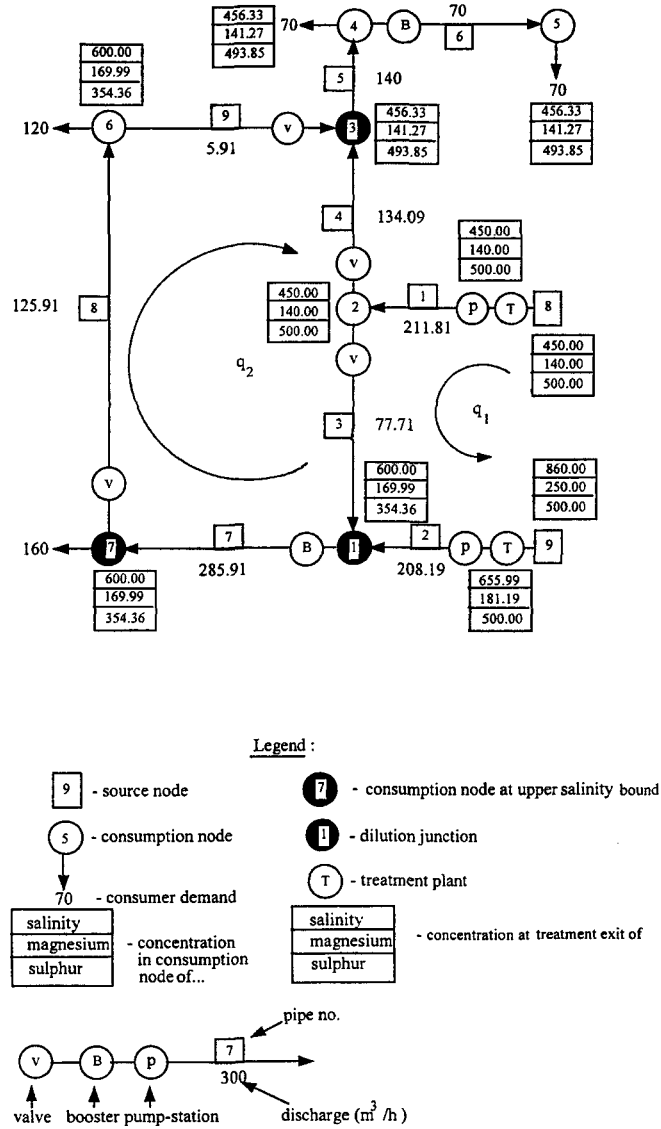
FIGURE 4 Optimal distribution of flows and water quality parameters.

pumping stations consist of 28 pumps, 14 control values, 14 aggregated consumers and 7 treatment plants located at the sources, where the product water is pumped into the distribution system. Water

TABLE VI   Optimal operation of pump stations

Pump station	Configuration	Head (m)	Flow (m^3/h)	Head loss (m)	By-pass (m^3/h)
A	axaxaxb	57.27	211.81	0.43	8.19
B	d	63.39	233.40	0.11	0.069

*The head loss is due the control valve.

TABLE VII   Pressure heads at the consumer nodes

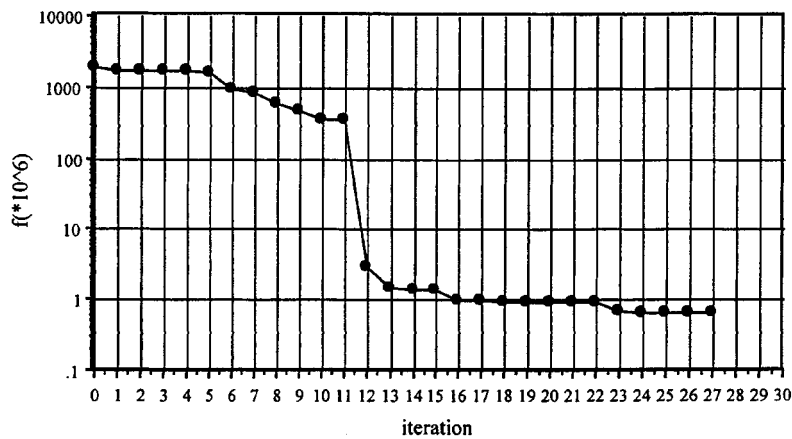| Node | Elevation (m) | Demand (m^3/h) | Pressure head (m) | | |
			min.	max.	optimal
4	− 258	70	40	80	49.68
5	− 217	70	40	80	40.00
6	− 242	120	35	80	39.31
7	− 246	160	40	80	57.87

FIGURE 5   The values of the objective function during the solution process.

TABLE VIII   Optimal operation of the boosters

Booster	On pipe	Flow m^3/h	Head m	Power kW
1	6	70.00	33.59	17.07
2	7	285.91	21.52	27.28

quality in this system is expressed by 3 parameters: salinity, magnesium, and sulphur, all of which are assumed to be conservative, i.e., they do not decompose or interact, and only dilute by flow and

TABLE IX   Optimal operation of the control valves

Valve	On pipe	Flow m^3/h	Opening ratio %	Head loss m
1	3	77.71	100	0.604
2	4	134.09	97.79	1.859
3	8	125.91	100	1.585
4	9	5.91	100	0.003

mixing. 4 nodes are supplies to domestic consumers and the others to agriculture. For the domestic consumers the quality parameters are restricted by upper limits, defined by water quality standards. At the agricultural nodes, net income and reduction yield functions reflecting the sensitivity of crops are defined. The other quality parameters at these nodes are restricted by upper limits. The solution for the Central Arava Network was obtained after 50 iterations in 75.76 sec on an IBM-3081 computer.

SUMMARY

This paper presents the full flow-quality-head (Q-C-H) model for operating a multi-quality water supply system under a single loading. The model combined the two models (Q-C and Q-H) of the previous papers in this series.

The characteristics of the Q-C-H model are:

• The objective is minimization of the total cost which includes cost of water at the sources, pumping, treatment and loss of income due to low water quality.

• The dilution conditions and the performance of each pumping station are smoothed.

• The problem is decomposed into an inner-outer structure and the inner problem is decomposed into a flow-quality Q-C and a flow head Q-H model in which the Q-H model is solved by SLP. The outer problem is solved by the projected gradient method combined with the Complex method.

An example network is solved for demonstration. The solution was achieved in 27 iterations when the operating cost reduced from $2 * 10^9$ to 647108 NIS.