OPTIMAL OPERATION
OF MULTIQUALITY WATER SUPPLY
SYSTEMS-III: THE Q-C-H MODEL

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A new technique for optimal operation of multiquality water supply systems is proposed. The technique, which is known as a Q-C-H (flow-quality-head) model, combines previously developed flow-quality (Q-C) and flow-head (Q-H) models for optimal operation of water supply systems. The decision variables in the model are the operation of treatment plants, pumps and valves. The model minimizes the cost of water at sources, treatment, energy, and loss of agricultural yield when water quality is low. The model uses an iterative modified projected gradient method combined with the Complex method. As in the Q-C and Q-H models, the solution method is based on decomposition, disaggregation/aggregation approach, involving internal and external optimization. The decision variables of the external model are the flows in the loops of the network and the removal ratios at the treatment plants. The operation of the pumps and valves are the decision variables of the internal model. The method is demonstrated by application to an example problem.

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INTRODUCTION

Optimal operation of multiquality networks has been discussed in the previous papers of this series [4, 5]. Several earlier attempts to develop

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models for this problem were somewhat limited. Sinai et al. [17] suggested decomposition of the complete solute transport problem which include 3 variables (Q-flow, C-concentration of water quality parameters, and H-head) into two simpler problems: Q-C ("chemical") and Q-H (hydraulic). They also suggested transformation of the Q and C variables to a solute flow variable $u = QC$ and to solve for it. The result is a limited Linear Programming (LP) model with hydraulic constraints, which can be used in networks where the dilution junction equation contains only one outflow unknown [17]. Mehrez et al. [11] used the same transformation of Sinai et al. [17] and developed a model for the optimal operation of multiquality networks. Their model is more advanced than that of Sinai et al. [17], yet it is also quite limited. This model is unable to simulate networks where the flow direction can be changed during the solution procedure. The same limitation arises in Mehrez et al.'s model [11]. In addition, it does not include treatment plants, and several constraints on C and H are missing. Ostfeld and Shamir [13] developed a more advanced model for the same problem. They used some of the mathematical tools developed by Cohen [2] and formulated a problem which was solved by a general software package GAMS/MINOS. This optimization technique does not make use of the properties of the problem, and therefore the utilization of this method is again limited.

In the present [2–5] the optimization problem is tackled by decomposition of the Q-C-H problem into two subproblems Q-C and Q-H. The two submodels are combined in this paper into a general Q-C-H model, using the solution methods developed earlier as building blocks.

**Parameters of Water Quality and Consumer Types**

Water quality may be described by primary and dependent quality parameters. Primary parameters are defined as those which are independent of other parameters. Dependent parameters are defined by a relationship between independent parameters, for example, SAR which is defined as the ratio between the concentration of Na, Mg and Ca in the water.
Water is supplied to 3 types of consumers:

1. Agricultural consumers, who have a crop yield function which depends on water quality [4].
2. Domestic and industrial consumers who do not require water treatment and do not incur costs from variable water quality at the consumer's connection.
3. Consumers with limits on the concentration of one or more water quality parameters. Such consumers require that quality be within specified limits, and no cost or benefit function for quality is specified.

**Mathematical Representation of the Network**

The water supply system can be described as a graph, consisting of arcs connected at nodes. See Cohen et al. [4] for a topological description of the network, and Cohen et al. [5] for definition of paths and the path matrix.

**Flow Distribution in the Network**

The optimal flow distribution is determined as part of the solution in the following manner. An initial flow distribution, which satisfies water continuity at all nodes, is specified. This flow distribution is modified by the solution process, in a way in which continuity is retained. This maintenance of continuity is achieved by considering the circular flows, \( q_i \), in loops and pseudo-loops, as decision variables, since these flows are modified the continuity at nodes is maintained. As a consequence of this definition, the water flow continuity equations can be omitted, thus reducing the size of the optimization model. The vector \( \mathbf{q} \) is of order \( n_1 \), the number of loops, with the \( i \)th component of the vector being the flow in the positive direction of loop \( i \). Since the number of loops is considerably smaller than the number of pipes, using the circular flows in the loops rather than the flow in the pipes as the decision variables in this manner results in an even smaller model.
The relationship between the flows in all pipes of the network, \( q_a \), are defined by:

\[
q_a = q_a^0 + L_a^T q
\]

(1)

where \( q_a^0 \) is the initial flow distribution, which satisfies the water flow continuity equations, and \((\cdot)^T\) denotes transpose. The discharges to be supplied from the sources are defined by:

\[
q_s = \hat{A} q_a = \hat{A} [q_a^0 + L_a^T q]
\]

(2)

where \( \hat{A} \) is a submatrix of \( A \), obtained from those rows of \( A \) which relate to source nodes. The discharges through the valves, boosters, treatment plants and pump stations are:

\[
q_k = B_k^T q_a = B_k^T [q_a^0 + L_a^T q] \quad k = p, b, t, v
\]

(3)

**Hydraulics of the Network Components**

See Cohen et al. [5] for the equations of pipes, valves and pumps. A special method is developed to describe pump stations, as follows:

The head-discharge-efficiency function (operating surface) of a pump station depends on the particular set of pumps (called "a configuration"), which are in operation and their series-parallel interconnections. If these configurations are the decision variables, then the result is an Integer Nonlinear Programming problem, which is difficult or impossible to solve for a system of practical size. Cohen et al. [5] have shown that a pump station can be used more efficiently if the configuration also includes a control valve and by-pass. They also describe an alternative formulation resulting from the inclusion of a valve and bypass for the operating surface of a pump station. The head-discharge relationship is defined by a smooth continuous function over a domain bounded by an envelope curve, \( h_{n}^{*} (q_p) \) (see Eq. (9) below), which is a limit on the head that can be supplied by the pump station as a function of its discharge. Each head discharge point under or on this envelope is defined with its efficiency, the least cost configuration for providing it, and the head which the control valve in the configuration has to dissipate. For a given \( q \) the efficiency as a function of the
head is expressed as:

\[ \eta = \sum_{i=1}^{n} \eta_i h_i \frac{\exp\{\varphi(\zeta_i - 1)\} + \exp\{\varphi(\zeta_i - 0.5)\}}{\exp\{\varphi(h_i - h_j)\} + \exp\{-\varphi(h_i - h_j)\}} \]  

(4)

where \( \varphi(\cdot) \) is a normalized function defined by:

\[ \varphi(x) = k_p \frac{x}{\sqrt{x^2 + \epsilon_0}} \]  

(5)

\( h_i \) is the head obtained by configuration \( i \) when the discharge is \( q \), and \( \zeta_i \) is defined by:

\[ \zeta_i = \sum_{j=1}^{i} \frac{\exp\{\varphi(h_i - h_j)\}}{\exp\{\varphi(h_i - h_j)\} + \exp\{-\varphi(h_i - h_j)\}} \]  

(6)

Pump stations usually consist of different pumps. Consequently, the envelope curve includes several nonsmooth points, which are smoothed as follows. Assume a pump station with \( n_i \) nonsmooth points \( q^i (i = 1, 2, \ldots, n_i) \); each interval between two such points being derived from a different envelope function; denote by \( g'(q) \) the function for the interval \( (q^i, q^{i+1}) \). The smoothed envelope function is given by:

\[ H_p(q) = \frac{\sum_{i=1}^{n} g_i(q) \exp(\varphi(\xi_i))}{\sum_{i=1}^{n} \exp(\varphi(\xi_i))} \]  

(7)

where:

\[ \xi_i = (q - q^i)(q^{i+1} - q) \]  

(8)

The \( (q = 0, h = 0) \) point also has to be included. Inclusion of this point is achieved by bringing the envelope curve down to this point for \( q < 1 \) (an arbitrary value, considered small enough). Thus, the smoothed envelope curve, including the operating condition at zero flow, is:

\[ H_p(q) = H_p(q_d) \frac{\exp(\varphi(q - 1)) + \alpha \exp(-\varphi(q - 1))}{\exp(\varphi(q - 1)) + \exp(-\varphi(q - 1))} \]  

(9)

where \( \alpha \) is a coefficient which determines how steeply the curve is brought down to zero.
Cohen et al. [5] have shown that the solutions derived on the basis of this transformation of the discrete operating surface into a continuous and smooth one are practical and effective.

FORMULATION OF THE HYDRAULIC SUBPROBLEM

For a given flow distribution in the network, the power at each booster is expressed as a quadratic function of the discharge, and heads are computed with the Hazen-Williams equation (5, Eq. (12)). The maximum head in each pump station, $h_p^m$, is given by Eq. (9) for the known flow. The pumping head can vary in the range $[0, h_p^m]$. Each valve causes a head loss, even when fully open, denoted by $(h_v^r)^0$ (5, Eq. (14)).

Thus, for a given flow distribution the operation of the boosters is known and the optimal operation of the pump station and control valves, which realize the given flow, is formulated in the following $Q_0$-$H$ problem [5]. The decision variables of this $Q_0$-$H$ problem are: $h_p^r$ – pumping heads at the $n_p$ pump stations, $h_v^r$ – the additional head losses to those of fully open valves in the $n_v$ control valves. Thus, the heads in valves are determined from:

$$h_v = h_v^r + (h_v^r)^0 \quad \text{where } h_v^r \geq 0 \quad (10)$$

The constraints consist of the continuity of energy (Kirchoff’s second law) and the head limits at nodes. Since there may be no feasible solution for the specified flow distribution in order to assure a mathematical solution, a pair of artificial variables with opposite signs, which are penalized in the objective function by coefficients $\theta^+$ and $\theta^-$, respectively, are added to each constraint. Denote by $x_p^+$ and $x_p^-$ the artificial variables added to the Kirchoff’s law constraints, and denote by $x_v^+$ and $x_v^-$, the artificial variables added to the head constraints.

The $Q_0$-$H$ optimization problem now becomes:

$$\min f_0 = tk_c \alpha_p \lambda_p^T Q_p H_p E^{-1} I_p + \theta^+ (x_p^+ + x_p^-) + \theta^- (x_v^+ + x_v^-) \quad (11)$$

where $t$ is the time period, $k_c$ is the cost per unit energy, $I_p$ is a vector of order $n_p$ whose elements are 1, $\alpha_p$ is a constant whose value depends
on the units used (for SI units \( \alpha_p = 104 \)), \( Q_p \) is a square diagonal matrix whose components \( q_p \) are the discharges at the \( n_p \) pump stations. \( H_p \) is a square diagonal matrix whose components \( h_p \) are the pumping heads at the \( n_p \) pump stations. \( E \) is a square diagonal matrix whose components \( \eta_p \) are the efficiencies.

**Continuity of Energy (Kirchoff’s Second Law)**

\[
L_p h_p - L_h h'_p + x'_p + x''_p = b_p + L_a h'_a + L_v (h'_v)^0 - L_h h'_h
\]  

(12)

where \( b_p \) is the head difference between the end of the loops: \( b_p = 0 \) when the loop is closed, and for a pseudo-loop \( b_p \) equals the known head difference between the ends of the pseudo-loop.

**Head Limits**

The head at each consumption node must be between a lower and an upper limit. The head at each consumption node is expressed by the energy equation along the path between the consumer node and a reference node at which the head is known independent of the flows (e.g., a reservoirs or well)

\[
h'_c - P_a h'_a + P_v (h'_v)^0 - P_b h'_b \leq P_p h_p - P_v h'_v + x'_p + x''_p
\]

\[
\leq h''_c + P_a h'_a + P_v (h'_v)^0 - P_b h'_b
\]

(13)

where \( h'_c \) and \( h''_c \) are the differences between the head at the reference node and the lower and upper allowable heads at the consumer, respectively.

**Pumping Head Limits**

At each pump station the pumping head is restricted to be less than the maximum specified at the given flow.

\[
H_p \leq h'_p
\]

(14)

with \( h'_p \) given by Eq. (9).
Non-negativity

\[ h_p, h'_p, x_L^+, x_L^-, x_p^+, x_p^- \geq 0 \] (15)

This Q-Ho problem has a nonlinear objective function and linear constraints. The nonlinearity in the objective function is caused by the dependency of the efficiency of a pump on the pump head \( h_p \). However, this nonlinearity is normally mild over a wide range of \( h_p \). For given \( \eta_p \), all the multipliers of \( h_p \) in the first term of Eq. (11) are therefore combined into a constant \( \rho^T \), resulting in a LP problem with the objective function:

\[ \min f_0 = \rho^T h_p + \theta^+(x_L^+ + x_p^+) + \theta^-(x_L^- + x_p^-) \] (16)

where \( \rho^T = tk_{p} \alpha_p Q_p E^{-1} \).

The Qo-H problem is solved by what is known as the Algorithm Qo-H, described by Cohen et al. [5]. The convergence of the Qo-H algorithm has not been proven mathematically. However, the algorithm has been applied over a wide range of cases and found to always converge in at most 3 iterations (Cohen [2]). The solution of this Qo-H problem gives the optimal pumping head, \( h_p^* \), and the optimal head loss in valves, \( h_v^* \), for the given flow distribution. The optimal opening ratio of each valve is obtained from Eq. (9) when \( h_v^* \) and the known discharge are substituted. The optimal operation of each pump station to achieve the optimal pumping head \( h_p^* \) is obtained from Algorithm OPP, described by Cohen et al. [5], when \( h_p^* \) and the given flow are used. The optimal operation of each pump station determined in this manner includes the pumps to be operated and their series-parallel interconnections, the head loss dissipated by the control valve and the by-pass flow.

MATHEMATICAL MODEL OF THE Q-C-H PROBLEM

The network flow problem which includes all three variables, \textit{i.e.}, flow (Q), concentration of water quality parameter (C) and head (H) distribution is what is referred to as the Q-C-H problem. This problem
was decomposed into two subproblems: "chemical" (Q-C) and "hydraulic" (Q-H) ones which were the subject of the two previous papers in this series [4, 5]. The complete model of the Q-C-H problem makes use of some elements of the Q-C and Q-H models. These two submodels are connected by the flow (Q) distribution in the network. Solving the two submodels independently does not yield the same optimum as that of the combined Q-C-H model since there are some interactions in the solute transport problem that should be described in a complete Q-C-H model. The following is an attempt to present a mathematical model for optimizing the steady state operation of multi-quality water supply network where all the three variables Q, C, H are considered.

Decision Variables

The primary decision variables of the Q-C-H problem are: q – the circular flows; r – the removal ratios in the treatment plants (n subvectors of primary quality parameters, where r is the removal ratios of the primary quality parameter m at the treatment plants (see [2–4]); c – the values of the primary parameters in the system (n subvectors of primary quality parameters, where c is the concentration of the quality parameter m at the nodes); c – the values of the dependent quality parameters in the system (n subvectors of dependent quality parameters, where c is the concentration of the dependent quality parameter m at the nodes).

The secondary decision variables are: the opening ratios of valves, configurations to be operated within the pump stations, head losses in control valves, and the by-pass flows. Recall that when q is known, these variables can be defined uniquely and directly from the solution of the Q-H problem.

Objective Function

The objective is to minimize total cost of operation of the system and losses incurred by reduction in productivity caused by lower water quality over a specified time period t. The total cost is made of the following components.
**Water Supply Cost**

This is the cost of water supplied from the sources. In general, the specific cost (per unit volume) varies with discharge. Thus, the specific cost of water at sources is given by a vector of functions, denoted by \( \mathbf{w}_s(\mathbf{q}_s) \), whose dimension is the number of sources. When the unit cost at a source is fixed, the value of the function for that source is a constant. The total supply cost from the sources, \( \phi_s \), for the entire period \( t \), is:

\[
\phi_s = t\mathbf{w}_s(\mathbf{q}_s)^T \mathbf{q}_s
\]  

(17)

**Pumping Cost of Boosters**

The total pumping cost of boosters, \( \phi_b \), for the time period is:

\[
\phi_b = tk_c \alpha_b \mathbf{N}_b(\mathbf{q}_b)^T \mathbf{l}_b
\]  

(18)

where \( k_c \) is the cost per unit energy, \( \alpha_b \) is a constant which depends on the units used (for SI units \( \alpha_b = 104 \)), \( \mathbf{N}_b(\mathbf{q}_b) \) is the power-discharge functions, and \( \mathbf{l}_b \) is a vector of order \( n_b \) whose elements are all 1.

**Pumping Cost of Pump Stations**

As described above, for a given flow distribution, defined by the circular flow, \( \mathbf{q} \), the pumping cost at pump stations is obtained from the solution of the \( Q_0-H \) problem, which is denoted by \( \phi_p(\mathbf{q}) \).

**Treatment Cost**

The total treatment cost, \( \phi_t \), is [4, Eq. (6)]:

\[
\phi_t = t\mathbf{w}_t(\mathbf{q})^T \mathbf{q}_t
\]  

(19)

**Yield Reduction Cost**

The total loss due to yield reduction, \( \phi_y \), is [4, Eq. (10)]:

\[
\phi_y = y_0^T B_0 [\mathbf{I}_y - \mathbf{y}]
\]  

(20)
where \( y \) and \( y_0 \) are vectors with components \( y_i \) and \( (y_0)_i \), respectively, and \( B_0 \) is the income matrix associated with full productivity with diagonal element \( i \) equal to \( (b_0)_i \).

**Penalty Costs**

Some consumers have limits on the concentration of water quality parameters they can accept or use. These limits on water quality are introduced into the objective function by a penalty function, such that the solution is directed to be within the specified limits. For quality parameter \( m \) in node \( i \), \( c_{mi} \) for which the limits are denoted by \( c'_{mi} \) and \( c''_{mi} \), the penalty function is defined by:

\[
P_{mi} = -z_{mi} \exp \{ \varpi(z_{mi}) \}
\]

where

\[
z_{mi} = (c_{mi} - c''_{mi})(c_{mi} - c'_{mi})
\]

and \( \varpi(\cdot) \) is the "normalized function" as defined in Eq. (5). Cohen et al. [3, 4] showed that this function is convex, continuous and smooth and can be controlled to prevent numerical difficulties often caused by penalty functions.

The penalty cost with respect to the primary parameter \( m \), \( \phi^n_L \), is:

\[
\phi^n_L = \sum_{i \in N} P_{mi}(c_{mi}) = \sum_{i \in N} -z_{mi} \exp \{ \varpi(z_{mi}) \} \quad N = N_1 + N_2 + N_3
\]

where the summation is only used over the nodes at which the value of the quality parameter \( m \) is restricted between limits. Similarly, the upper and lower limits, \( c_{dni} \) and \( c'_{dni} \), which are imposed on the dependent parameter \( m \) in node \( i \) are introduced in the objective function by a penalty term defined by:

\[
P_{mi} = -e_{mi} \exp \{ \varpi(e_{mi}) \}
\]

where

\[
e_{mi} = (c_{dni} - c''_{dni})(c_{dni} - c'_{dni})
\]
The penalty cost with respect to the dependent parameter $m$, $\phi_D^p$, is:

$$\phi_D^p = \sum_{i \in N} P_{ni}(c_{dm}) = \sum_{i \in N} -e_{mi} \exp\{\varpi(e_{mi})\} \quad N = N_1 + N_2 + N_3 \quad (26)$$

again with the summation only occurring across those nodes where the value of the water quality parameter $m$ is restricted.

Combining the components mentioned above yields the objective function, $f$:

$$\min f = rw_s(q_s)^T q_s + rw_t(r_t)^T q_t + t k_x \alpha_b N_b(q_b)^T 1_b +$$
$$+ \phi_p(q) + y_i^T 1_y [y - y] + \sum_{m=1}^{n_3} \phi_L^m + \sum_{m=1}^{n_3} \phi_D^m \quad (27)$$

**Constraints**

The operator $\phi_p(q)$, which defines the solution of the problem $Q_0-H$, includes the hydraulic constraints: Kirchoff’s second law, limits on the heads at consumers', limits on the pumping head at pump stations and limits on the opening ratio at valves. The additional constraints of the problem $Q-C-H$ consist of the mass conservation at nodes, the dilution conditions, the relations between the dependent and primary parameters, limits on the discharges in various components of the network, and limits on the removal ratios at the treatment plants.

**Mass Conservation Law**

This law is expressed by $n_3$ (number of primary quality parameters) sets of equations. Each set includes equations for all nodes, except the source nodes.

$$\sum_{j \in E_1} a_{yj} q_{j_m}^l c_{jm}^l + \sum_{i \in E_2} a_{yi} q_i^l (1 - \bar{r}_m) c_{jm}^l - d_i c_{im} = 0 \quad \forall i \in N_2 \ & \forall m \in M_1 \quad (28)$$

where $d_i$ is the consumption at node $i$, $M_1$ is the primary quality parameters group, $a_{yj}$ is the element $ij$ of the adjacency matrix $A$, $c_{jm}^l$ is the concentration of primary quality parameter $m$ in pipe $j$, which is
defined by the smoothed dilution conditions [3, 4]:

$$c_{im}^+ = \frac{c_{km}\exp\{\varpi(a_0q_{a_0}^i)\} + c_{wm}\exp\{-\varpi(a_0q_{a_0}^w)\}}{\exp\{\varpi(a_0q_{a_0}^i)\} + \exp\{-\varpi(a_0q_{a_0}^w)\}}$$  \hspace{1cm} (29)

where $k$ and $i$ are the end nodes of pipe $j$. This formulation overcomes the difficulty encountered by Shah and Sinai [15, 16] in specifying the dilution conditions, by allowing the flows in each pipe to reverse direction during the iterations of the solution process as the optimal flow distribution is identified. $r_m$ is the removal ratios of the primary parameter $m$ at pipes belonging to $E_2$, defined by:

$$\tilde{r}_m = B_f^2 r_m$$  \hspace{1cm} (30)

**Quality Parameter Relation Function**

According to the definition of the dependent parameter group, each dependent water quality parameter has a function which defines its relationship with primary parameters. An example is SAR, which depends on the concentrations of Ca, Na and Mg. These functions are incorporated into the constraints:

$$c_{\text{dim}} = x_{\text{im}}(c_{i1}, c_{i2}, \ldots, c_{im}) \quad \forall i \in N_1 \text{ } \& \text{ } \forall m \in M_2$$  \hspace{1cm} (31)

where $M_2$ is the dependent parameter group, $x_{\text{im}}$ is the relation between dependent parameter $m$ and the primary parameters at node $i$.

**Limits on Discharges**

**Boosters**

The discharges through the boosters are limited:

$$0 \leq q_b \leq q_b^u$$  \hspace{1cm} (32)

where $q_b = \text{booster flow}$.

**Valves**

Limits on the discharge through the valve are imposed to prevent cavitation, and to prevent reversal of flow if the valve is a one-way
valve.

\[ q'_v \leq q_v \leq q''_v \]  \hspace{1cm} (33)

where \( q''_v \) = maximum flow through valves.

For a one-way valve \( q'_v = 0 \), and for a bi-directional valve \( q'_v = -q''_v \).

**Pump Stations**

The maximum discharge in each pump-station, \( q''_p \), is obtained from the envelope curve, defined in Eq. (9):

\[ 0 \leq q_v \leq q''_p \]  \hspace{1cm} (34)

Using Eq. (3) constraints (26–28) transform into:

\[ q'_k - B_k^T q^0_0 \leq B_k^T L_q q \leq q''_k - B_k^T q^0_0 \quad \text{with } k = b, v, p \]  \hspace{1cm} (35)

for boosters, one-way valves and pump stations \( q'_b = q'_v = q'_p = 0 \).

**Sources**

The upper limit, \( q''_s \), is either the supply capability of the source or some other restriction. The lower limit, \( q'_s \), is to prevent reverse flow or to prevent overflow if reverse flow into a reservoir occurs. The constraints at all sources are:

\[ q'_s \leq q_s \leq q''_s \]  \hspace{1cm} (36)

Using Eq. (2) this translates into:

\[ q'_s - \hat{\mathbf{A}} q^0_0 \leq \hat{\mathbf{A}} L_q q \leq q''_s - \hat{\mathbf{A}} q^0_0 \]  \hspace{1cm} (37)

**Limits on Removal Ratios**

\[ r' \leq r \leq r'' \]  \hspace{1cm} (38)

where \( r' \) and \( r'' \) are lower and upper limits on the removal ratios at the treatment plants for each primary water quality parameter. According to the definition of the removal ratio \( r'' \leq 1 \).
OPTIMIZATION STRATEGY

For a given flow distribution in the network and given removal ratios in the treatment plants the water quality values throughout the network are defined by Eq. (22), which becomes linear. The values of the dependent parameters can then be obtained from Eq. (31). The decision variables can thus be divided into two groups: the first contains the control variables, \( u \), which consist of the vectors \( q \) and \( r \); the second group, denoted by \( x \), is the distribution of water quality values throughout the network, and are called the state variables (or resultant variables). Using this definition of terms, the Q-C-H problem has the general form.

\[
\begin{align*}
\text{(Problem } P_0 \text{)} & \quad \min f_0(x, u) \\
\text{subject to:} & \\
& \quad g(x, u) = 0 \\
& \quad h' \leq h(u) \leq h''
\end{align*}
\]

\( f_0(x, u) \) is the objective function given by Eq. (27). \( g(x, u) \) are the mass conservation equations for the primary water quality parameters, including the dilution conditions with respect to the distribution of the quality parameters at nodes (Eqs. (28) and (29)). \( h(u) \) are the functions of the circular flows and of the removal ratios constrained between bounds \( h' \) and \( h'' \). Equations (35) and (37) are related to the circular flows \( q \), while the constraints denoted by Eq. (38) are related to the removal ratios \( r \).

The Problem \( P_0 \) can be transformed into the equivalent problem:

\[
\begin{align*}
\text{(Problem } P_1 \text{)} & \quad \min \Psi(u) = f_0(x(u), u) \\
\text{subject to:} & \\
& \quad h' \leq h(u) \leq h''
\end{align*}
\]

In other words, \( \Psi(u) \) is a function of \( u \) and of \( x \), such that \( x \) is obtained by solving the equations \( g(x, u) = 0 \), when \( u \) is given.

The Problem \( P_1 \) has linear constraints and a nonlinear objective function. The complexity of the Problem \( P_1 \) is however considerably
lower than that of the original problem, and therefore it is preferable to solve it.

PRINCIPLES OF THE SOLUTION METHOD

Problem $P_1$ can be solved using the projected gradient method. The main steps of the method are: (1) computation $\Psi(u)$; (2) computation of $\nabla_u \Psi$; (3) computation of the projected gradient if at the current $u$ there are active constraints; (4) computation of the modified direction of change; (5) updating $u$ if optimality conditions are not satisfied; and (6) using the Complex Method (Box, 1969 cited in [9]) when the algorithm gets “stuck” in a non-smooth point which is not optimal.

Computation of $\Psi(u)$

For a given $u = (q, r)$ the flows in all network arcs are known, and the costs of supply, treatment and pumping at boosters can be computed from Eqs. (17), (18) and (19), respectively. The pumping cost at the pump stations is obtained by solving the problem $Q_0-H$, as described by Cohen et al. [3, 5].

The yield loss cost $\phi_y$, and the penalty costs $\phi_L$ and $\phi_P$ depend on the water quality distribution which is found by solving the mass conservation equations ($g(x, u) = 0$). When $q$ and $r$ are given these equations are linear. Furthermore, they can be decomposed into one set for each water quality parameter. That is, for each water quality parameter there is a separate linear system of equations, all with the same coefficient matrix but with different right hand vectors. As a result, when $q$ and $r$ are given, the equation system $g(x, u) = 0$ has the following form:

$$Kc^m = b^m \quad m = 1, 2, \ldots, n_3$$  \hspace{1cm} (43)

where $c^m$ is the distribution of quality parameter $m$ at nodes. $K$ is the specific quality discharge matrix whose components are computed by:

$$K_{ij} = \begin{cases} d_{ij}Q_{ij} & \text{if } d_{ij}Q_{ij} \geq 0 \text{ and } i \neq j \\ -\sum_k d_{ik}Q_{ik} + d_i & \text{if } d_{ij}Q_{ij} < 0 \end{cases}$$  \hspace{1cm} (44)
$Q_{ij}$ is the flow from node $i$ to $j$, $d_{ij}$ is element $ij$ of the matrix $D$, and $b^m$ is a vector of the mass flow input of primary parameter $m$, whose components are computed by:

$$(b^m)_i = \sum_{j \in N_i} Q_{ji} c^m_j$$  \hspace{1cm} (45)$$

in which $Q_{si}$ is the discharge from source node $s$ to node $i$, and $c^m_s$ is the concentration of parameter $m$ at source node $s$.

Since each system of equations has the same matrix, solution by $LU$ decomposition is preferable. That is, the matrix $K$ is first decomposed into a lower triangular matrix $K_L$ and an upper triangular matrix $K_u$ such that $K = K_L K_u$. The distribution of the primary parameter $m$ is obtained by solving

$$K_L w^m = b^m$$  \hspace{1cm} (46)$$

for $w^m$ and

$$K_u c^m = w^m$$  \hspace{1cm} (47)$$

for $c^m$. Since the matrices $K_L$ and $K_u$ are triangular, $w^m$ and $c^m$ can be obtained by forward and back substitution, respectively. The distribution of the dependent quality parameters are computed by the functions defined in Eq. (31). They yield loss cost $\phi_y$ and the penalty costs $\phi^a$ and $\phi^b$ are computed from Eqs. (20), (23) and (26), respectively.

**Computation of the Gradient**

The gradient of $\Psi(u)$ with respect to the control variables $u$, $\nabla_u \Psi$, is:

$$\nabla_u \Psi = \nabla_u f + [\nabla_u g]^T \beta$$  \hspace{1cm} (48)$$

where $[\nabla_u g]$ is the Jacobian of Eq. (40a) with respect to $u$, and the vector $\beta$ is computed from:

$$[\nabla_x g]^T \beta = -\nabla_x f$$  \hspace{1cm} (49)$$

$\nabla_u f$ consists of two subvectors $\nabla_q f$ and $\nabla_x f$:
\n\$\nabla_u f = [\nabla_q f \nabla_x f]^T$

where:

$$\nabla_q f = tF_eB_e[q^0_e + L_e q]$$  \hspace{1cm} (50)$$
in which \( F_r \) is the Jacobian of \( w_\alpha(r) \) with respect to \( r \). The matrix is square and diagonal in which element \( ii \) is:

\[
(F_r)_{ii} = \frac{dw_i}{dr}
\]  

(51)

\( \nabla_q f \) is defined by:

\[
\nabla_q f = tL_\alpha \hat{A}^T [F_\alpha \hat{A} (q^p_\alpha + L_\alpha q) + w_\alpha] \\
+ tL_\alpha B_\alpha w + c_{b} k_c L_b F_b 1_b + \nabla_q \phi_p
\]  

(52)

where \( F_a \) and \( F_b \) are the Jacobian matrices of \( w_\alpha \) and \( N_\beta \) with respect to \( q_\alpha \) and \( q_\beta \), respectively. Both are square diagonal, with components defined, respectively, by:

\[
(F_a)_{ii} = \frac{dw_i}{dq_i}
\]  

(53)

\[
(F_b)_{ii} = \frac{dN_i}{dq_i}
\]  

(54)

\( \nabla_q \phi_p \) is the gradient of the operator \( \phi_p(q) \) which yields the optimal solution of the problem Q_0-H. Cohen et al. [3, 5] have shown that

\[
\nabla_q \phi_p = \left[ L_a \alpha_a L_a^T - L_b \alpha_a L_b^T + L_c \alpha_c L_c^T \right] \pi_L \\
+ \left[ L_a \alpha_a P_a^T - L_b \alpha_a P_b^T + L_c \alpha_c P_c^T \right] \pi_p \\
+ L_c \pi_c \alpha_c \left\{ I_p - Q_p E_p^{-1} (S_{q_1} + S_{q_2}) \right\} E_p^{-1} \eta_p
\]  

(55)

where \( \pi_L \), \( \pi_p \) and \( \pi_c \) are the dual variables associated with Eqs. (12), (13) and (14), respectively.

\( S_a \), \( S_b \), \( S_c \) are square diagonal matrices of order \( n_a \), \( n_b \), \( n_c \) and \( n_p \), respectively. Their components are defined by:

\[
(S_a)_{ii} = 1.852 k_a^l (d_a)^{0.852}
\]  

(56)

\[
(S_c)_{ii} = \alpha_c^l k_c^l (q_c^l)^{\alpha_c - 1}
\]  

(57)

\[
(S_b)_{ii} = d_{b_p}/dq_p
\]  

(58)

\[
(S_c)_{ii} = (d\theta)/dq_p
\]  

(59)
$I_p$ is a unit matrix of order $n_p$, $S_{q_1}$ and $S_{q_2}$ are the Jacobian matrices of the pumping efficiency with respect to $q_p$ and $h_p$, respectively. Both are square diagonal matrices of order $n_p$ with components:

$$(S_{q_1})_{ii} = d h_p / dq_p$$

$$(S_{q_2})_{ii} = d h_p / dh_p$$

In the second term in Eq. (48), $[\nabla u]$ is computed as follows. $[\nabla u]$ is the Jacobian matrix of Eqs. (28) with respect to $u$. It consists of $n_3$ pairs of submatrices; each is of order $n_m$, one is a derivative with respect to $q$ and the other with respect to $r$, where submatrix $m$ is related to the primary parameter $m$. The vector $\beta$ is computed by solving Eq. (40a).

The matrix $[\nabla x]$ is block diagonal, where the matrix block $m$, $[\nabla^m x]$, is related to primary parameter $m$. Furthermore, the submatrix is the specific input mass flow matrix, $K$, and therefore the vector $\beta$ can be computed in separate parts:

$$K^T \beta^m = -\nabla^m x f \quad \forall m \in M_1$$

where $\beta^m$ is subvector of $\beta$ related to primary parameter $m$. $\nabla^m x f$ is derived from the gradient of the yield loss cost and penalties with respect to the primary parameter $m$:

$$\nabla^m x f = \nabla^m x \phi_y + \nabla^m x \phi_L + \nabla^m x \left( \sum_{j=1}^{n_c} \phi_j^L \right)$$

$$= \sum_{j=1}^{n_c} \phi_j^L$$

where $F_y$ is the Jacobian matrix of the relative yield function with respect to the primary parameter $m$ among the consumer nodes. Cell $im$ contains the derivative of the relative yield function at node $i$ with respect to primary parameter $m$:

$$(F_y)_{im} = -b_{q} dy_{q} / dc_{m}$$

(60)

(61)

(62)

(63)

(64)
$F^m_y$ is the column of the matrix $F_y$ related to primary parameter $m$. The matrix $F_D$ is related to the dependent parameters and is defined similar to the matrix $F_z$:

$$(F_D)_{im} = -b^l_0 y^l_0 \frac{dy^i}{dc^m_l}$$

(65)

$F_z$ is the Jacobian matrix of the dependent parameters with respect to the primary parameters. The matrix consists of $n_3$ submatrices. Submatrix $m$ is defined with respect to primary parameter $m$ whose cell $kl$ is:

$$(F_z)_{kl} = \frac{dX_{kl}}{dc^m_l}$$

(66)

$F_z$ is the Jacobian of the penalty function $\phi_L$ with respect to primary parameters. Cell $im$ contains the derivative of $\phi_L$ with respect to primary parameter $m$:

$$(F_z)_{im} = \frac{dP_{pm}}{de^i_m}$$

$$= -\exp\{\varpi(z_{im})\} \left[ 1 - \frac{k_p z_{im} \varepsilon}{z_{im}^2 + \varepsilon} \right] [e''_{im} + e'_m - 2e_{im}]$$

(67)

in which $z_{im}$ is computed from Eq. (22) according to $e_{im}$ which is obtained from Eq. (28) for the given $q$ and $r$.

$F_L$ is the Jacobian of the penalty function $\phi_D$ with respect to the independent parameters, and is defined similar to the matrix $F_z$:

$$(F_L)_{im} = \frac{dP_{im}}{dc_{dm}^i}$$

$$= -\exp\{\varpi(e_{im})\} \left[ 1 - \frac{k_p e_{im} \varepsilon}{e_{im}^2 + \varepsilon} \right] [e''_{dim} + e'_{dim} - 2e_{dim}]$$

(68)

in which $e_{im}$ is computed from Eq. (25).

Since $K$ in Eq. (62) has been decomposed into $K_L$ and $K_u$, the subvector $\beta^m$ can be computed simply by solving

$$K_u^T \omega^m = -\nabla^m f$$

(69)
for \( w^m \) and then \( \beta^m \) is computed from

\[
K_L^T \beta^m = w^m
\]  

(70)

The matrices \( K_u \) and \( K_f \) are triangular, and thus \( w^m \) and \( \beta^m \) can be obtained by back and forward substitution, respectively.

**Computation of the Projected Gradient**

Following Rosen [14], the projected gradient, \( s_u \), is:

\[
s_u = -\left[ \nabla_u \Psi + N_a^T \lambda \right]
\]

(71)

in which \( N_a \) is the coefficient matrix of the active constraints, and \( \lambda \) is obtained from

\[
N_a^T N_a \lambda = -N_a^T \nabla_u \Psi
\]

(72)

In the present problem \( N_a \) has the form

\[
N_a = A_a \begin{bmatrix} 0 \\ 0 \\ J_e \end{bmatrix}
\]

where \( A_a \) is the coefficient matrix of the active set related to \( q \), and \( J_e \) is the coefficient matrix of the active set related to \( r \). \( \nabla_u \Psi = \left[ \nabla_q \Psi; \nabla_r \Psi \right] \), thus the system described by Eq. (72) can be decomposed into two subsystems:

\[
A_a A_a^T \lambda_1 = -A_a \nabla_q \Psi
\]

(73)

\[
J_e J_e^T \lambda_2 = -J_e \nabla_r \Psi
\]

(74)

where \( \lambda_1 \) and \( \lambda_2 \) are sub-vectors of \( \lambda \). Since \( J_e J_e^T = I \) then the projected gradient with respect to \( r \), \( s_r \), is

\[
\begin{align*}
s_r^i &= 0 & \text{if } i \in I_r, \\
\end{align*}
\]

\[
\begin{align*}
s_r^i &= -\nabla_r \Psi & \text{if } i \notin I_r.
\end{align*}
\]

(75)

where \( I_r \) is the active set with respect to \( r \).

There are several methods for computing \( s_q \), the projected gradient with respect to \( q \). Cohen, et al. [3, 4] show that the projected gradient
can be computed directly as follows. First, the matrix \( A_qA_q^T \) is decomposed into \( A_L \) and \( A_U \) such that \( A_qA_q^T = A_LA_U \), and then \( \lambda_1 \) is computed by solving

\[
A_Lw = -A_q\nabla_q\Psi
\]

for \( w \) and \( \lambda_1 \) is obtained from

\[
A_U\lambda_1 = w
\]

by back and forward substitution, respectively. Substituting \( \lambda_1 \) into Eq. (71) yields

\[
s_q = -[\nabla_q\Psi + A_q^T\lambda_1]
\]

As long as the active set does not change, no recomputation of \( A_L \) and \( A_U \) is needed, and the vector \( \lambda_1 \) is computed for each new \( \nabla_q\Psi \). The matrices of \( A_L \) and \( A_U \) are updated if a constraint is added to the active set, and a new decomposition is required only when the active set is reduced, which occurs only when the current \( \|s_q\| \) equals zero. This aspect of the algorithm reduces the computational burden of the model significantly.

Computation of the Modified Projected Gradient

The removal ratios at the treatment plants are in the range \([0, 1]\), whereas the circular flows can be of the order of tens, hundreds and even thousands. Furthermore, the effect of a unit change in a removal ratio is much greater than the effect of a unit change in the circular flow. As a result, there is a scaling problem between \( r \) and \( q \). Methods for nonlinear optimization are in general sensitive to scaling, and especially so the projected gradient which is an extension of steepest decent. Gill et al. [8] (pp. 273–275) have suggested transformation of the variables as a method to overcome these difficulties. In this problem transformation can be performed by defining \( r \) in percent (%) when the circular flows are in the order of hundreds or in parts per thousand (‰) when the circular flows are in the order of thousands. However, attempts to apply this approach to the problem failed, and the computational difficulties remained. The alternative approach used in the paper is to note that, for a separable quadratic problem, the
optimal solution is achieved in one iteration, independent of the initial point and the order of the scaling. As shown above, $s_r$ and $s_q$ are orthogonal. Therefore the problem is both quadratic and separable with respect to $q$ and $r$, and the search direction can be found by the following two-phase algorithm.

**Algorithm TF (Two Phases)**

**Step a** Assume initial search directions $d'_q = s_q$ for $q$, and $d'_r = s_r$ for $r$.

**Step b** Compute $\alpha''_q$, the maximal step along $d'_q$, and $\alpha''_r$, the maximal step along $d'_r$, such that $(q + \alpha''_q d'_q)$ and $(r + \alpha''_r d'_r)$ do not violate a constraint which is not included in the active set.

**Step c** Compute $\alpha'_q$, the optimal step length along $d'_q$:

$$\Psi(q + \alpha'_q d'_q, r) = \min \{ \Psi(q + \alpha d'_q, r) | 0 \leq \alpha_q \leq \alpha''_q \}$$

The computation of the optimal step length is discussed below.

**Step d** Compute $\alpha'_r$, the optimal step length along $d'_r$:

$$\Psi(q, r + \alpha'_r d'_r) = \min \{ \Psi(q, r + \alpha r d'_r) | 0 \leq \alpha_r \leq \alpha''_r \}$$

Note that the one dimensional search along $q$ and $r$ independently and separately remains on the manifold of the active set.

**Step e** Compute the norm of the updated direction, $s$:

$$\|s\| = \left[ (\alpha''_q (d'_q)^T d'_q + \alpha''_r (d'_r)^T d'_r) \right]^{1/2}$$

**Step f** Compute $d'_q$, the normalized modified projected gradient direction related to $q$:

$$s_q = \alpha'_q d'_q / \|s\|$$

**Step g** Compute $d'_r$, the normalized modified projected gradient direction related to $r$:

$$s_r = \alpha'_r d'_r / \|s\|$$

Note that $d_r = d'_r$ if $s_r = 0$ and $d_q = d'_q$ if $s_q = 0$. 


The optimal step length in steps \( c \) and \( d \) could be determined by one of the methods for one dimensional search, such as the Powell or Davidson methods (Avriel [1], pp. 221 – 240). When the optimality conditions are not met, a one dimensional search is performed along the updated direction, \( s \), such that an optimal step length is obtained. In some cases, the computation of \( \Psi \) and \( \nabla \Psi \) can be computationally intensive and therefore a non-exact search method is preferable. The efficiency of the scaling method is probably reduced for non-exact methods, since its efficiency for separable quadratic problem is derived for an exact one dimensional search. This was examined over a wide range of examples, and it was found that the overall efficiency of the algorithm is good.

**Incorporation of the Complex Method**

So far it has been shown that the Q-C-H problem can be formulated as a optimization problem in the subspace of \( q \) and \( r \) with a nonlinear objective function and linear constraints. For a given \( q \) and \( r \) the optimal operation of the pump stations and of the control valves are determined by solving the Q\(_{iq}\)-H problem. The gradient of the objective function is computed analytically as detailed above. Eiger et al. [6] showed that the objective function in such an “inner-outer” optimization problem may have non-smooth points. In the present problem this is due to the inclusion of \( \phi_{\rho}(q) \), the solution of the problem Q\(_{iq}\)-H. The objective function of the outer optimization is neither convex nor concave, with linear constraints. Thus, even the application of a non-smooth optimization method can not guarantee a solution.

The Complex Method was developed by Nelder and Mead [12] for unconstrained problems. Box (1969), cited in Jacoby et al. [9], extended it to problems with inequality constraints. The method is based on the objective function values and does not use gradients, so it is applicable to non-smooth problems. The efficiency of the method is improved when the feasible region is convex, since the search process is guaranteed to remain in the feasible domain. This is the case in the present problem since the constraints are linear. The efficiency of the Complex method is reduced substantially, however, as the number of decision variables increases. The modified projected gradient and the
Complex methods are therefore combined. The process is performed by the modified projected gradient while the projected gradient is effective, i.e., while it provides the descent direction. When the process approaches a non-smooth (kink) point the gradient may show incorrectly the descent direction, since the gradient is only one of the members of the subdifferential set. In this situation the one dimensional process terminates with a null step where the Kuhn-Tucker conditions are not satisfied. At this point the Complex method is used to reach a better point, with a search based only on function values.

**Termination Criteria**

When the relative change of the objective function at two successive points is less than a set small value \( \epsilon_r \) or a relative change in the decision variables in two successive iterations is less than another set small parameter \( \epsilon_w \), then the Kuhn-Tucker optimality condition is satisfied if the multipliers with respect to the active constraints at their upper bounds are nonnegative, and non-positive for the active constraints at their lower bounds. Otherwise, the active constraint having the multiplier with maximal absolute value is dropped from the set, and the solution process continues with respect to the updated active set. However, it is possible that the current point is non-smooth and then the Kuhn-Tucker conditions are not true for the optimal point. Thus if the subdifferential set consists of a null projected subgradient the current point is optimal, otherwise the Complex method is used to reach a better point, with a search based only on function values. If no improved point is found, then the current point is the best that can be found (from the initial point).

**Algorithm Q-C-H**

**Initialize**  
Assume initial value \( u^0 := \{q^0, r^0\} \). Initialize the iteration counter \( k := 0 \).

**Step 1**  
Compute the distribution of the primary quality parameters by solving:

\[
K e^n = b^n
\]
Step 2 Compute the distribution of the dependent quality parameter by using the relation functions:

\[ c_{dm} = \chi_{nm}(c_{n1}, c_{n2}, \ldots, c_{nm}) \quad \forall n \in N_3 \quad \text{and} \quad \forall m \in M_2 \]

Step 3 Solve the Q-D-H problem at the current flow distribution \( q^k \).

Step 4 Compute the value of the objective function, \( \Psi(u^k) \).

Step 5 Compute \( \nabla_q f \) from Eq. (63).

Step 6 Compute \( \beta^i \) by solving:

\[ K^T \beta^m = -\nabla_q^m f \quad \forall m \in M_1 \]

Step 7 Compute \( \nabla_q f \) and \( \nabla f \) from Eqs. (52) and (50), respectively.

Step 8 Compute \( \nabla_q \Psi \) and \( \nabla_q \Psi \) from:

\[ \nabla_q \Psi = \nabla_q f + [\nabla_q g]^T \beta \]
\[ \nabla_q \Psi = \nabla_q f + [\nabla_r g]^T \beta \]

Step 9 Compute \( \lambda_1 \) by solving:

\[ A_L A_D \lambda_1 = -A_a \nabla_q \Psi \]

Step 10 Compute \( \lambda_2 \) from \( \lambda_2 = -\nabla_r \Psi \)

Step 11 Compute the projected gradient with respect to \( q \) from:

\[ s_q = -[\nabla_q \Psi + A_L^T \lambda_1] \]

and with respect to \( r \) from:

\[ s_r = 0 \quad \text{if } i \in I_r \]
\[ s_r = -\nabla_r \Psi \quad \text{if } i \notin I_r \]

Step 12 Compute the modified projected gradient as described in Algorithm TF.

Step 13 Compute the optimal step length along the modified projected gradient, \( \alpha^* \).

Step 14 Check the optimality conditions. If satisfied goto Step 16, otherwise goto Step 15.
Step 15 Updating: If the active set has changed, update the active matrix \( A_g \) and the decomposition matrices \( A_L \) and \( A_U \). Update \( q \) and \( r \) from \( q^{k+1} := q^k + \alpha \cdot d_q \) and \( r^{k+1} := r^k + \alpha \cdot d_r \), update the iteration counter: \( k := k+1 \), and return to Step 1.

Step 16 Complex search: Use the Complex procedure. If an improved point is found, go back to Step 0 with the improved point as the initial flow distribution, otherwise, go to Step 17.

Step 17 Compute the optimal operation of the valves with \( q_v \) and \( h_v \) at the current point substituted. \( q_v \) is defined from Eq. (3) according to \( q^k \), and \( h_v \) is obtained from the solution of the problem Q-H0 at the current point.

Step 18 Compute at each pump station the optimal configuration, the optimal head loss dissipated by the pumping control valve and the optimal by-pass flow by using the algorithm OPP, described by Cohen et al. [3, 5] where \( q_p \) and \( h_p \) are considered. Note that \( q_p \) is defined from Eq. (3) according to \( q^k \), and \( h_p \) is obtained from the solution of the problem Q0-H at the current point.

EXAMPLE

The technique described in the previous sections was applied to the network shown schematically in Figure 1. The network consists of 9 pipes, 9 nodes, 2 pump stations, 2 boosters, and 4 control valves and is fed from two constant head reservoirs at nodes 8 and 9, and delivers to consumers at nodes 4, 5, 6 and 7. The network is operated 2000 hours and the cost per unit energy is 0.22 NIS/kWh (S1US = 3.6 NIS, NIS = New Israeli Shekel).

The water quality parameters of importance in this system are salinity, magnesium and sulphur. The relevant quantity and quantity data for the sources are given in Table I.

At the consumption nodes 4, 5 and 6 a quadratic yield function of the following form was defined with respect to salinity \( y = a_0 + a_1 c + a_2 c^2 \), in which \( y \) is the relative yield, \( c \) is the salinity in mg-cl/l, and \( a_0, a_1 \) and \( a_2 \) are constants as shown in Table II. At node 7 salinity is
TABLE I Data for sources

<table>
<thead>
<tr>
<th>Node</th>
<th>Specific cost (NIS/m³)</th>
<th>Elevation* (m)</th>
<th>Maximal discharge (m³/h)</th>
<th>Salinity (mg/l)</th>
<th>Magnesium (mg/l)</th>
<th>Sulphur (mg/l)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.638</td>
<td>-252.5</td>
<td>325</td>
<td>450</td>
<td>140</td>
<td>500</td>
</tr>
<tr>
<td>9</td>
<td>0.256</td>
<td>-255.0</td>
<td>700</td>
<td>860</td>
<td>250</td>
<td>300</td>
</tr>
</tbody>
</table>

*Elevations are negative because the system is in the Jordan Valley, below sea level.
TABLE II  Data for consumption nodes with respect to quality

<table>
<thead>
<tr>
<th>Node</th>
<th>( b_0 \times y_0 )</th>
<th>Coefficients of the yield function</th>
<th>Upper limits for water quality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \times 10^{-8} )</td>
<td>( a^0 )</td>
<td>( a_1 \times 10^5 )</td>
</tr>
<tr>
<td>4</td>
<td>0.700</td>
<td>1.0</td>
<td>1.250</td>
</tr>
<tr>
<td>5</td>
<td>0.448</td>
<td>1.0</td>
<td>-3.06</td>
</tr>
<tr>
<td>6</td>
<td>1.200</td>
<td>1.0</td>
<td>1.250</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Note: \( b_0 \times y_0 \) is the income when there is no yield loss due to water quality.

TABLE III  Hydraulic data for consumption nodes

<table>
<thead>
<tr>
<th>Node</th>
<th>Consumption ( m^3/h )</th>
<th>Elevation m</th>
<th>Minimum pressure head m</th>
<th>Maximum pressure head m</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>70</td>
<td>-258</td>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>-217</td>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>6</td>
<td>120</td>
<td>-242</td>
<td>35</td>
<td>80</td>
</tr>
<tr>
<td>7</td>
<td>160</td>
<td>-246</td>
<td>40</td>
<td>80</td>
</tr>
</tbody>
</table>

constrained by an upper bound. Upper limits as shown in Table II were defined for the other water quality parameters.

Other relevant data for the consumption nodes are given in Table III.

Treatment plants for salinity and magnesium are placed on pipe 2, and a treatment plant for sulphur is placed on pipe 1. The cost with respect to treatment (removal) of salinity is \( w_s = 2.151 \times 10^{-4} r^2 \) and with respect to magnesium and sulphur is \( w_r = 1.32 \times 10^{-4} r^2 \) in which \( w_r \) is the specific cost in NIS/m\(^3\), and \( r \) is the removal ratio in %. All the removal ratios are limited to 75%.

Identical control values with the following head loss function were placed on pipes 3, 4, 9 and 8. \( h_v = 10^{-6} q_v^2 t^{-1.5} \), in which \( h_v \) is the head loss in m, \( q_v \) is the discharge in m\(^3\)/h and \( t \) is the opening ratio. Boosters, each with cubic head-discharge function, are placed on pipes 6 and 7. \( h_b = a_0 + a_1q_b + a_2q_b^2 + a_3q_b^3 \) and linear power-discharge function \( N_b = b_0 + b_1q_b \) where \( h_b \) is the head in m, \( N_b \) is the power in kW, and \( q_b \) is the discharge in m\(^3\)/h. The values of the coefficients \( a_i (i=0,\ldots,3) \) and \( b_i (i=0,1) \) for the head discharge and power discharge functions are given in Table IV.

The data for pipes are given in Table V. All pipes have \( Chw = 120 \).
The network consists of 2 loops: one closed loop of pipes 3, 7, 8, and 9, and one pseudo-loop between the nodes 8 and 9 which includes pipes 1, 2, and 3. The initial flows, the positive directions of the pipe flows and of the circular flows are shown in Figure 1.

The pump station placed on pipe 1 (pump station \( A \)), consists of 3 pumps of type \( a \) and one pump of type \( b \). The pump station placed on pipe 2 (pump station \( B \)), consists of 2 pumps of type \( c \) and one pump of type \( d \). The head-discharge and efficiency-discharge functions for the four types of pipes are given in Figures 2 and 3. Only parallel connections between the pumps are possible.

The optimal solution is \( f^* = 647108.83 \text{ NIS} \), and the optimal circular flows are \( q_1^* = -88.19 \text{ m}^3/\text{h} \), \( q_2^* = -14.09 \text{ m}^3/\text{h} \) (flows relative to those in Fig. 1). The optimal distributions of flows and quality parameters are given in Figure 4.

The optimal removal ratios of salinity and magnesium in the treatment plants placed on pipe 2 are 23.72\% and 25.72\%, respectively. The optimal removal ratio of sulphur in the treatment plant placed on pipe 1 is zero. The optimal operation of the pump station and the pressure heads at the consumer nodes are summarized in Tables VI and VII, respectively.

The solution was obtained after 27 iterations in 4.39 sec on an IBM-3081 computer. The values of the objective function during the solution process are shown in Figure 5. The results are presented in Tables VI–IX.
The method was also applied to the water distribution system of Central Arava Region, which supplies an area in the southern region of Israel. This system has 38 nodes, 39 pipes, 11 sources (wells), 9
pumping stations consist of 28 pumps, 14 control values, 14 aggregated consumers and 7 treatment plants located at the sources, where the product water is pumped into the distribution system. Water
TABLE VI  Optimal operation of pump stations

<table>
<thead>
<tr>
<th>Pump station</th>
<th>Configuration</th>
<th>Head (m)</th>
<th>Flow (m³/h)</th>
<th>Head loss (m)</th>
<th>By-pass (m³/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>axaxaxb</td>
<td>57.27</td>
<td>211.81</td>
<td>0.43</td>
<td>8.19</td>
</tr>
<tr>
<td>B</td>
<td>d</td>
<td>63.39</td>
<td>233.40</td>
<td>0.11</td>
<td>0.069</td>
</tr>
</tbody>
</table>

*The head loss is due to the control valve.

TABLE VII  Pressure heads at the consumer nodes

<table>
<thead>
<tr>
<th>Node</th>
<th>Elevation (m)</th>
<th>Demand (m³/h)</th>
<th>Pressure head (m)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>min.</td>
<td>max.</td>
</tr>
<tr>
<td>4</td>
<td>258</td>
<td>70</td>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>217</td>
<td>70</td>
<td>40</td>
<td>80</td>
</tr>
<tr>
<td>6</td>
<td>242</td>
<td>120</td>
<td>35</td>
<td>80</td>
</tr>
<tr>
<td>7</td>
<td>246</td>
<td>160</td>
<td>40</td>
<td>80</td>
</tr>
</tbody>
</table>

FIGURE 5  The values of the objective function during the solution process.

TABLE VIII  Optimal operation of the boosters

<table>
<thead>
<tr>
<th>Booster</th>
<th>On pipe</th>
<th>Flow (m³/h)</th>
<th>Head (m)</th>
<th>Power (kW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>70.00</td>
<td>33.59</td>
<td>17.07</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>285.91</td>
<td>21.52</td>
<td>27.28</td>
</tr>
</tbody>
</table>

quality in this system is expressed by 3 parameters: salinity, magnesium, and sulphur, all of which are assumed to be conservative, i.e., they do not decompose or interact, and only dilute by flow and
mixing. 4 nodes are supplies to domestic consumers and the others to agriculture. For the domestic consumers the quality parameters are restricted by upper limits, defined by water quality standards. At the agricultural nodes, net income and reduction yield functions reflecting the sensitivity of crops are defined. The other quality parameters at these nodes are restricted by upper limits. The solution for the Central Arava Network was obtained after 50 iterations in 75.76 sec on an IBM-3081 computer.

SUMMARY

This paper presents the full flow-quality-head (Q-C-H) model for operating a multi-quality water supply system under a single loading. The model combined the two models (Q-C and Q-H) of the previous papers in this series.

The characteristics of the Q-C-H model are:

- The objective is minimization of the total cost which includes cost of water at the sources, pumping, treatment and loss of income due to low water quality.
- The dilution conditions and the performance of each pumping station are smoothed.
- The problem is decomposed into an inner-outer structure and the inner problem is decomposed into a flow-quality Q-C and a flow head Q-H model in which the Q-H model is solved by SLP. The outer problem is solved by the projected gradient method combined with the Complex method.

An example network is solved for demonstration. The solution was achieved in 27 iterations when the operating cost reduced from $2 \times 10^9$ to 647108 NIS.
References