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OPTIMAL OPERATION OF RESERVOIRS BY STOCHASTIC PROGRAMMING

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A model for optimal multi-period operation of a multi-reservoir system with uncertain inflows and water demands is formulated and solved by the Finite Generation Algorithm. Uncertainties are considered in chance constraints and in penalties due to deviations from meeting demand and reservoir level targets. The penalty functions are linear-quadratic, can be imposed on deviations in one or both directions from the target, and are easily fitted to data by selection of parameters. The stochastic variables are assigned discrete probability distributions. The primal (optimal operation) problem is solved by formulating the dual and then finding its optimum (which is proven to be global for the conditions specified) via a sequence of linear-quadratic deterministic optimization problems of controlled size. The method is demonstrated for a three-reservoir two-period problem. Sensitivity analysis with respect to parameter values is presented. Stochastic simulation is used, to augment the information given by the optimal solution.

KEY WORDS: Reservoirs, operations, stochastic programming

INTRODUCTION

The reservoir management problem addressed in this paper is defined as follows:

—There exists a system of interconnected reservoirs, such as that shown in Figure 1.

—The system is to be operated over a number of time periods (e.g., years, seasons, months).

—We wish to determine releases from all reservoirs and withdrawals from the rivers for all time periods, in advance.

—Constraints exist on maximum and minimum reservoir levels and on maximum admissible flows in the rivers.

—The inflows are stochastic, with known probability distributions. The inflows may be dependent or independent random variables. In this paper we assume the latter. Joint distributions cause added computational effort.

—Target values may be set for some or all of the reservoir levels at selected points in time over the planning horizon. For example, a minimum low level target may be desired for the end of the dry season, and/or a maximum high level target for the wet season. Maximum admissible exceedence probabilities may also be specified for these levels and are converted into chance constraints.

—Penalties can be imposed on deviations (in either one or both directions) from these target levels.

—The demands which are to be supplied by withdrawals from the rivers are also stochastic, with known probability distributions. As with inflows, only the case of independent distributions is presented in this paper. Allowing joint distributions is possible, at increased computational cost.

—Target values can be set for the stochastic demands, and maximum allowed probabilities for dropping below them imposed. In addition, a penalty may be associated with shortages.

—A benefit is associated with the flows (the decision variables), for example releases from a reservoir through a hydropower plant or supply to customers. The benefit for each decision variable is quadratic and concave (diminishing returns to scale).

—The objective function is the sum of all benefits over the planning horizon minus the expected value of all losses (penalties).

—The optimization model is cast as a stochastic program with recourse. The decisions for all time periods are selected ("here-and-now") such that the stated constraints for all time periods are satisfied (with probability, where appropriate) and such that for any possible outcome of the stochastic variables of the first period there exists a feasible recourse ("wait-and-see") action. The nature of this recourse action is not considered explicitly; its consequences are represented in the model by the penalty function. This function can be viewed as the optimal value of the recourse for each possible deviation from the target (Yeh[18] p. 1800).

—Only the decisions for the first period are executed. Then one waits for the random variables of the first period to materialize, takes recourse actions for the first period, and again formulates and solves the management problem for the new planning horizon, with the actual initial conditions observed.

Similar reservoir management problems have been addressed in the past. In his state-of-the-art paper Yeh[18] discusses these. We shall discuss here only previous work which deals with reservoir operation problems close to the one we solve. Dupacova's work[2] is probably closest. The problem she formulates is quite similar to ours, with two main differences: (1) the objective function is linear, while ours is linear-quadratic, and (2) the penalty functions are piece-wise linear, while ours are linear-quadratic. Dupacova develops a network representation of the problem, but does not provide a solution method for the stochastic problem. There is reference to some earlier methodological work by Rockafellar and Wets on stochastic programming with simple recourse but the paper does not contain its application.

Hicks *et al.*[4] use a non-linear programming technique to solve a reservoir operation problem in which "soft" constraints are converted into penalties on stochastic deviations from targets. Hogan *et al.*[5] survey approaches to formulation and solution of decision problems under risk, in particular chance constrained programming and stochastic programming with recourse. They conclude that chance-constrained programming limits the way in which the decision maker must state his attitude to risk, and therefore the usefulness of the technique is restricted. As for models using stochastic programming with recourse, Hogan *et al.*[5] state that their solution is difficult and costly.

Yeh[18] provides a comprehensive review of the field. Since that time there have been several publications which deserve mention. Wang and Adams[17] describe the hydrological inflows as a Markov process and find the steady state and the immediate real-time operating policies by a method they call generalized policy iteration. Their formulation and approach are quite different from ours. Takeuchi[14] uses chance constrained programming for management of a single reservoir, with particular emphasis on drought conditions. Strycharczyk and Stedinger[13] provide added criticism of chance constrained programming models for reservoir management. They state that the method is limited, and that it overestimates required reservoir volumes, i.e., it is wasteful. Trezos and Yeh[15] use a stochastic dynamic programming method to solve a problem quite similar to the one we address in this paper. The system dynamics of their problem are linear, the objective is quadratic, and penalties may be added for deviations from stated targets.

Yeh[18] states:

"Extensive literature review of the subject of optimization of reservoir operations reveals that no general algorithm exists. The choice of methods depends on the characteristics of the reservoir system being considered, on the availability of data, and on the objective and constraints specified."

Publications since 1985 do not change this conclusion. We now propose to add another approach to formulation and solution of a reservoir management problem, based on a stochastic optimization method developed recently by Rockafellar and Wets[9,10]. We believe that this method allows the decision maker to introduce his attitude to risk in a more satisfactory way than was hitherto possible, while the model can still be solved for reasonably sized problems. Somlyody and Wets[12] have already demonstrated the application of this method to the management of lake eutrophication.

PROBLEM FORMULATION

Notations

E	Mathematical expectation
$\sim$	Indicates a stochastic quantity
x_j	Quantity per period at segment j in the network
c_j, r_j	Benefit function parameters
$\mathbf{v}_k$	Deviation from target at point k
p_{ik}, q_{ik}	Penalty function parameters
s_j	Upper bound on x_j
a_{ij}	Entries of the matrix A of flow continuity constraints
O_{ij}	Entries of the matrix O for the "hard" deterministic constraints
b_i	Right hand side vector of the "hard" constraints block
t_{kj}	Entries of the matrix T of stochastic "soft" constraints
$\mathbf{h}_k$	Target quantity at point k

Consider the three reservoir system in Figure 1. The natural inflow into each reservoir is stochastic. A power house is installed on each reservoir, and the flow through it can be controlled. When the reservoir overflows and spills occur they by-pass the power station and enter the stream. Water is withdrawn from the stream for consumption. The remaining flows from the two upstream reservoirs enter the downstream one. The remaining flow in the lowest branch of the system is used to meet another demand.

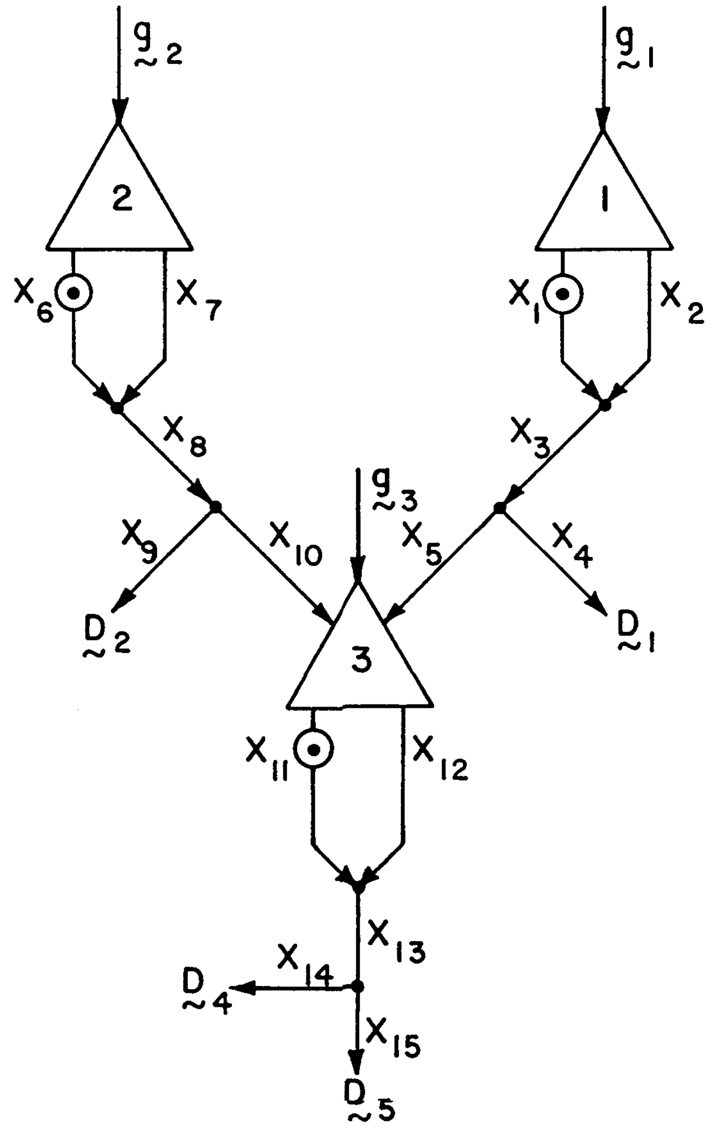

Figure 1 The reservoir system.

For clarity of presentation we begin with an annual operation model, which will later be expanded into a multiperiod model. The annual inflows into the reservoirs, $\tilde{g}_j$ and the demands, $\tilde{D}_j$, are random variables with known probability distributions. The initial water storages in the reservoirs, LI_j, are known. A target final volume, LF_j, may also be given. This final storage will not necessarily be met, because it will depend on random inflows as well as on our decisions. The decision variables are the flows (actually volumes of water) in all river branches.

The optimal operation problem is defined as follows. Given: initial storages, target final storages, probability functions for inflows and for demands, benefit and/or cost functions for the decision variables, penalties for not meeting demands and final target storages; find values of the decision variables which maximize the expected value of the objective function. These decisions are to be executed "here-and-now", i.e. before the random variables are realized.

The objective function is:

$$\text{Max } f(x) = \sum_{j=1}^{n} [c_j x_j - 1/2 r_j x_j^2] - E\left[\sum_{k=1}^{K} \Gamma_k(\tilde{v}_k; p_{1k}, p_{2k}, q_{1k}, q_{2k})\right] \tag{1}$$

The first part is deterministic, quadratic and separable. By requiring $r_j > 0$ we have a concave function, showing diminishing returns with scale compatible with economic theory. The second part is the expected value of penalties which will be incurred if prescribed targets of demands and future reservoir levels are not met. Denoting by TR_k the kth target, the deviation from it will be:

$$\tilde{v}_k = \sum_{j=1}^{n} t_{kj} x_j - \tilde{h}_k \tag{2}$$

where $\tilde{h}_k$ is the random component in the equation which defines TR_k. The deviation $\tilde{v}_k$ incurs a penalty whose functional form is:

$$\Gamma_k(\tilde{v}_k) = \begin{cases} -q_{2k}\tilde{v}_k - \frac{1}{2}p_{2k}q_{2k}^2 & \text{if} \quad \tilde{v}_k \leq -q_{2k}p_{2k} \\ \frac{1}{2}\tilde{v}_k^2/p_{2k} & \text{if} \quad -q_{2k}p_{2k} \leq \tilde{v}_k \leq 0 \\ \frac{1}{2}\tilde{v}_k^2/p_{1k} & \text{if} \quad 0 \leq \tilde{v}_k \leq q_{1k}p_{1k} \\ q_{1k}\tilde{v}_k - \frac{1}{2}p_{1k}q_{1k}^2 & \text{if} \quad q_{1k}q_{1k} \leq \tilde{v}_k \end{cases} \tag{3}$$

with $p_{1k}, p_{2k}, q_{1k}, q_{2k} > 0$, for $k = 1, \ldots, K$.

This function is shown in Figure 2. It is quadratic for small deviations and linear for larger ones. Its four parameters allow considerable flexibility in fitting it to data. The quadratic part may be extended for the entire range of probable deviations, by selecting p to fit the data and then making (pq) large. The function can be made essentially linear by selecting q to fit the data and then making (pq) small. The quadratic term at the origin must be maintained, however small, since this is required for the development of the solution method. A one-sided penalty function may be specified by setting to zero the parameter q of the side which is to be removed.

As an example of such deviations, consider that LF_3 is a specified target volume in reservoir 3 at the end of the year, while the known initial volume in that reservoir

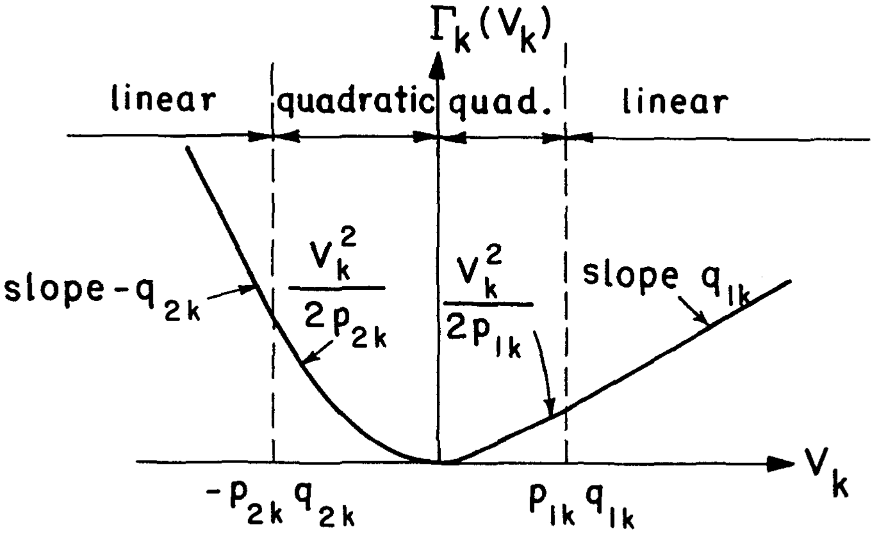

Figure 2 The penalty function.

is LI_3. Then the deviation from this target is

$$\underset{\sim}{v} = LF_3 - (LI_3 + x_5 + x_{10} + \underset{\sim}{g}_3 - x_{11} - x_{12}) \tag{4}$$

This is written in the form of (2) as:

$$\underset{\sim}{v} = LF_2 + (-x_5 - x_{10} + x_{11} + x_{12}) - (\underset{\sim}{g}_3 + LI_3) \tag{5}$$

where the random part is $\underset{\sim}{h} = \underset{\sim}{g}_3 + LI_3$.

The penalties caused by deviations from LF_3 may be due to flooding when the level is too high and/or to loss of recreational benefits when the level is too low.

As another example, the deviation from meeting the target demand $\underset{\sim}{D}_4$, which is stochastic, is:

$$\underset{\sim}{v} = x_{14} - \underset{\sim}{D}_4 \tag{6}$$

Here the penalty is incurred only when $x_{14} < \underset{\sim}{D}_4$, and supplemental water has to be brought from an outside source.

The decision variables are subject to several types of constraints:

a) Bounds, representing capacities of river branches:

$$0 \le x_j \le s_j \qquad j = 1, \ldots, n \tag{7}$$

b) Continuity equations:

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \qquad i = 1, \ldots, m_1 \tag{8}$$

These are "hard" constraints which are physical laws. For example $x_1 + x_2 + x_3 = 0$.

c) Deterministic equivalents of chance constraints:

$$\sum_{j=1}^{n} o_{ij}x_j \leq b_i \qquad i = 1, \ldots, m_2 \tag{9}$$

For example, if LF_3 is a desired target volume in reservoir 3 at the end of the year, it is required to reach at least this volume with probability of α (say 0.95), and the (known) initial volume in this reservoir is LI_3, then we require:

$$\text{Prob}[LI_3 + x_5 + x_{10} + \underset{\sim}{g}_3 - x_{11} - x_{12} \geq LF_3] = \alpha \tag{10}$$

This is converted into

$$LI_3 + x_5 + x_{10} + g_3^\alpha - x_{11} - x_{12} \geq LF_3 \tag{11}$$

where g_3^α is selected such that

$$\text{Prob}[\underset{\sim}{g}_3 \geq g_3^\alpha] = \alpha \tag{12}$$

Finally the constraint takes the form

$$-x_5 - x_{10} + x_{11} + x_{12} \leq LI_3 + g_3^\alpha - LF_3 \tag{13}$$

This concludes the presentation of the mathematical formulation for a single-period model. It is a stochastic programming problem with recourse. We decide on the x_j, which determine the deterministic part of the benefit. The random variables then materialize and deviations, $\underset{\sim}{v}_k$, from targets occur. Recourse actions are taken at this stage, for example purchase of water from an outside source to fulfill demands which have not been met by the x's, or payment of penalties for reservoir levels higher or lower than targeted for recreation or wildlife habitat. These actions result in a cost $\Gamma_k(\underset{\sim}{v}_k)$. For each possible realization of the random variables $\underset{\sim}{g}_j$ and $\underset{\sim}{D}_j$ the random deviations $\underset{\sim}{v}_k$ take on different values and result in certain penalties. We seek those decisions x_j which maximise the expected benefit given by (1).

The penalty function $\Gamma_k(\underset{\sim}{v}_k)$ shown in Figure 2 may itself be the optimal solution of a decision problem, namely the recourse problem, which we need not consider explicitly here. That decision problem is: what are the optimal decisions for a given set of deviations from the targets, deviations which are known once the random variables have materialized.

The multiperiod management problem is an extension of the single-period problem presented above. Consider Figure 3, where reservoir 1 is shown for two time periods denoted by the superscripts 1 and 2. For the second period the constraints on flows, continuity at nodes, and meeting demands are the same as for the first period (with superscript changed). Any hard or soft constraint on storage at the end of the second period is formulated with the sum over the two periods. For example:

$$0 \leq [LI_1^1 + (g_1^1 + g_1^2) - (x_1^1 + x_2^1) - (x_1^2 + x_2^2)] \leq LMAX \tag{14}$$

and

$$\underset{\sim}{v} = LF_1^2 - [LI_1^1 + (g_1^1 + g_1^2) - (x_1^1 + x_2^1) - (x_1^2 + x_2^2)] \tag{15}$$

For a multi-period model we must have constraints like (14) for each reservoir

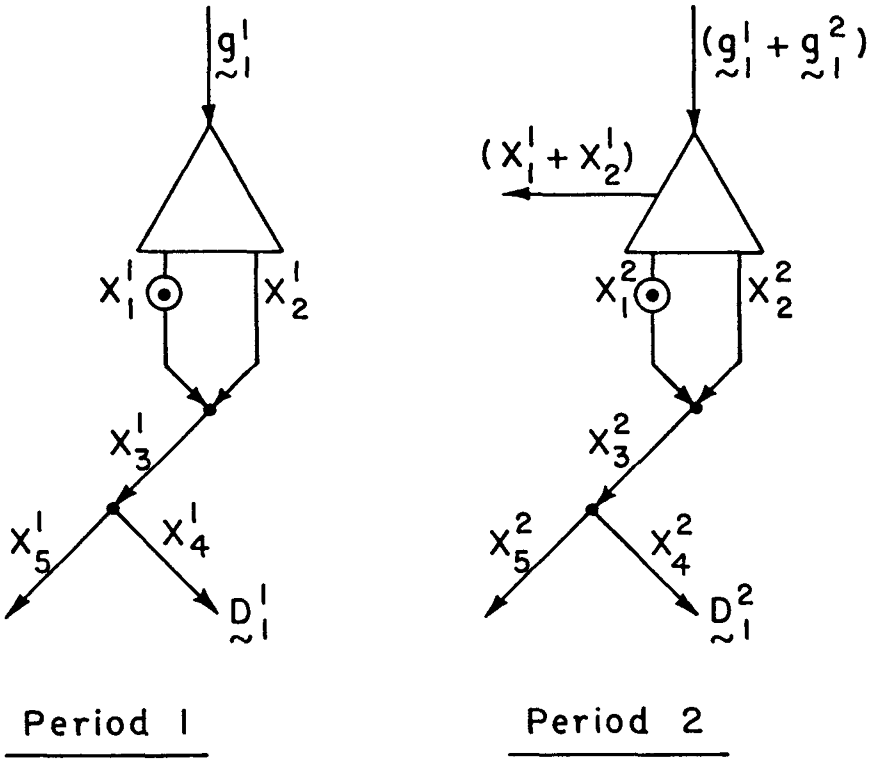

Figure 3 Reservoir 1. Two time periods.

for the end of each time period. Deviations defined like (15) are required only for those points in time (and locations) where target storages are specified. The structure of the constraint matrices for a multi-period model is presented in Appendix A.

In these constraints the random variable is the cumulative inflow from the initial time to the end of the time period in question. The probability distributions of these variables are obtained by direct analysis of the data for these summed flows, or by appropriate combination (considering dependence) of single period inflow probability distributions.

A multi-period model should be used in practice as follows:

1) Select the time horizon for planning the operation. For example this may be one year ahead, or a longer or shorter period such that its end is at a time when reservoir levels can reasonably be forecast, e.g., full at the end of winter.

2) Divide the planning horizon into periods. For example, the year may be divided into 12 months. Alternatively, the first period or two may be short, for example two weeks each, and the remainder of the horizon grouped into a few longer periods which coincide with hydrologic and water demand seasons.

3) Specify "soft" reservoir target constraints only at those reservoirs and times where they are important. For example, in a 12 month model only low level targets for end of summer (reliability of supply) and/or high level targets for winter (safety against floods) may be important. Set similar "soft" constraints for meeting demands, if appropriate.

4) Solve the model.

5) Execute the decisions for the first time period.

6) Observe events in the first period, update the data, formulate the model for a new planning horizon, solve, and so on.

METHOD OF SOLUTION

The stochastic program (P) defined by the ojective function (1) and the constraints (7)–(9), the deviations defined by (2), and the penalty function given in (3), is solved by the Finite Generation Algorithm (FGA) of Rockafellar and Wets[9,10]. Their algorithm was developed for a one-sided penalty function; the extension to a two-sided penalty was carried as part of our study[3].

The steps of the FGA are:

a) Formulate the dual (D), which is stochastic.

b) The dual is replaced by a sequence of deterministic quadratic programming sub-problems, whose feasible regions are polytopes contained in the feasible region of (D). The dimension of these problems is controlled and can be kept small (this is the "finite generation"). Under the conditions stated above for the primal (P) this sequence converges to the *global* optimum of (D), from which the optimal solution of (P) is obtained directly.

We shall present here only a concise formulation of the dual, without proofs. Readers interested in the theory are referred to Rockafellar and Wets[9,10].

For (P) we define the Lagrangian:

$$L(X, Y_1, Y_2, \underset{\sim}{Z}) = \sum_{j=1}^{n} [c_j x_j - \tfrac{1}{2} r_j x_j^2] + \sum_{i=1}^{m_1} y_{1i}\left[-\sum_{j=1} a_{ij} x_j\right] + \sum_{i=1}^{m_2} y_{2i}\left[b_i - \sum_{j=1}^{n} o_{ij} x_j\right] + \sum_{k=1}^{K} E\left\{\underset{\sim}{z}_k\left[\underset{\sim}{h}_k - \sum_{j=1}^{n} t_{kj} x_j\right] + \tfrac{1}{2}\underset{\sim}{p}_k \underset{\sim}{z}_k^2\right\} \tag{16}$$

with

$$\begin{aligned} X &= \{x_1, \ldots, x_n) | 0 \le x_j \le s_j\} \\ Y_1 &= \{y_1 = (y_{11}, \ldots, y_{1m_1})\} \\ Y_2 &= \{y_2 = (y_{21}, \ldots, y_{2m_2}) | y_{2j} \ge 0\} \\ \underset{\sim}{Z} &= \{\underset{\sim}{z} = \underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_k) | -q_{2k} \le \underset{\sim}{z}_k \le q_{1k}\} \end{aligned} \tag{17}$$

By general duality theory we obtain the dual

$$\text{minimize } g(Y_1, Y_2, \underset{\sim}{Z}) = \sum_{i=1} b_i y_{2i} + E\left\{\sum_{k=1}^{K} [h_k \underset{\sim}{z}_k + \tfrac{1}{2} p_k(\underset{\sim}{z}_k)^2]\right\} + \sum_{j=1}^{n} \Gamma_1(w_j; r_j, s_j) \tag{18}$$

s.t.:

$$\begin{aligned} y_{2i} &\geq 0 \qquad i = 1, \ldots, m_2 \\ -q_{2k} \leq \underset{\sim}{z}_k &\leq q_{1k} \qquad k = 1, \ldots, K \\ w_j = c_j - \sum_{i=1}^{m_1} y_{1i} a_{ij} &- \sum_{i=1}^{m_2} y_{2i} o_{ij} - E\left\{\sum_{k=1}^{K} \underset{\sim}{z}_k t_{kj}\right\} \end{aligned} \tag{19}$$

where:

$$\Gamma_1(w_j) = \begin{cases} 0 & \text{if } w_j \leq 0 \\ \tfrac{1}{2} w_j^2 / r_j & \text{if } 0 \leq w_j \leq r_j s_j \\ s_j w_j - \tfrac{1}{2} r_j s_j^2 & \text{if } w_j \geq r_j s_j \end{cases} \tag{20}$$

Note that $\underset{\sim}{z}_k$ are the only random variables in (D), and that they appear only in bound constraints.

The dual can be solved directly as a linear-quadratic problem, if we require each $\underset{\sim}{z}_k$ to have a discrete probability distribution, i.e., it has finitely many possible outcomes in the range $(-q_{2k} \leq \underset{\sim}{z}_k \leq q_{1k})$. Indeed, for small problems it is possible to solve the dual directly with each random variable "spread out" over the entire set of its possible values with their corresponding probabilities (Wagner[16]). For problems of more realistic size this is no longer feasible computationally and the solution must be obtained by the FGA method.

In order to study the nature of the solution obtained by the FGA we used a stochastic simulation model. Its structure, and the analysis performed on its results are discussed in the following section.

THE SIMULATION MODEL

The stochastic simulation model was written with SLAM II (Allan and Pritsker[1]). It was used for two purposes:

a) To study the results of the optimization, and perform sensitivity analyses with respect to values of the parameters, and

b) To augment the results of the optimization by providing (approximate) distributions of various resultant random variables, which may be useful to the decision maker.

Given values of all the decision variables, the simulation program generates a large number of sets of realizations for the random variables. For each set, all outcomes are computed: flows, overflows, supplies, shortages, final reservoir volumes for all time periods, as well as the value of the objective function and the penalties incurred. The sample of values for each such variable is then plotted as a histogram and analyzed to obtain statistics, such as the mean, variance, and probabilities of not meeting the stated targets. These values can be compared with the results of the optimization model.

The histograms and statistics of the outcomes can provide useful information. In particular, decision makers may be interested in the following:

a) Distribution of the objective function values. The optimization gives only the expected value, and the distribution may have some significance,

b) Distributions of the penalty values, in particular the extent of large penalties and their corresponding probabilities,

c) Distribution of the violation of "soft" constraints.

The example presented below will demonstrate the use of simulation as a complement to the stochastic optimization.

IMPLEMENTATION

The three reservoir problem presented above was solved for two seasons in a year. The model was formulated with GAMS (Kendrick and Meeraus[6]. GAMS (General Algebraic Model System) provides a concise and convenient tool for generating the optimization model as input to the optimization package. Solutions were computed with MINOS 5.0 (Mathematical In-core Non-linear Optimization System) (Murtagh and Saunders[8]), a general purpose non-linear optimization package.

All computations were carried out on a PC/AT with 640KB memory and an 8087 math co-processor. For a problem with 30 decision variables, 14 stochastic variables with 5 to 25 values on each distribution, running time was about 90 minutes. This is from a cold start, with no initial values prescribed. In a sequence of runs, as is the case in the sensitivity analysis presented below, and as would also occur in making runs for a real application, a better starting point is easily provided, and this greatly reduces running time. Our model is strictly quadratic ($r_j > 0$ for all j). The theory of the FGA guarantees convergence for this case at a linear rate. In our computations we indeed found very rapid convergence. For details see Eiger[3].

RESULTS

Table 1 includes data for the Base Run of the three-reservoir two-period model. There are 30 decision variables: 15 flows (quantities per period) for each period. Fourteen targets are specified: four demands $(D_1, D_2, D_4, D_5,)$ and three final levels $(LF_1, \ldots, LF_3)$ in each period. Probability distributions for the demands are given in Table 2. Probability distributions for the inflows are given in Table 3. For the second period the inflows (denoted by the superscript 2) are the summed flows for the two periods, their probabilities are computed by the appropriate combination of probabilities of the single period flows.

Table 1 Data for the Base Run.

Variable	*Parameter name*	*Value* (*)
Bounds on decision variables	$s_j \quad j = 1, \ldots, 30$	20
Initial storage in reservoirs	LI_1	10
	LI_2	8
	LI_3	6
Target storage levels, at the end of both periods	$\mathrm{LF}_1^1 = \mathrm{LF}_1^2$	10
	$\mathrm{LF}_2^1 = \mathrm{LF}_2^2$	8
	$\mathrm{LF}_3^1 = \mathrm{LF}_3^2$	6
Objective function coefficients (equal for all x_j)	$c_j \quad j = 1, \ldots, 30$	8
	$r_j \quad j = 1, \ldots, 30$	2
Penalty function coefficients	p_{1k}	0.2
	p_{2k}	0.2
	$q_{1k} \quad k = 1, \ldots, 14$	1
	q_{2k}	1
Prob. of violating chance constraints	α (see Eq. (14))	0.05

* All volumes are in 10^6 m^3

Table 2 Probability distributions of demands.

D_1^1		D_1^2		D_2^1		D_2^2		D_4^1		D_4^2		D_5^1		D_5^2	
val.	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*
0.5	0.02	1.0	0.02	1.0	0.05	2.0	0.05	2.0	0.05	1.0	0.05	4.0	0.10	4.0	0.10
0.7	0.06	1.2	0.06	1.3	0.10	2.8	0.10	2.5	0.10	1.5	0.10	4.5	0.15	4.5	0.15
0.9	0.30	1.4	0.30	1.6	0.40	3.6	0.40	3.0	0.40	2.0	0.40	5.0	0.25	5.0	0.25
1.0	0.50	1.6	0.50	1.9	0.25	4.2	0.25	3.5	0.25	2.5	0.25	5.2	0.30	5.2	0.30
1.2	0.12	1.8	0.12	2.2	0.20	5.8	0.20	4.0	0.20	3.0	0.20	5.4	0.20	5.4	0.20

In the Base Run we have used the same cost and penalty coefficients for all variables and targets, for simplicity of presentation and interpretation of the results. Below we shall discuss results of sensitivity analyses in which we examined the effects of changing various parameters. The changes are shown in Table 4. Six series of sensitivity runs were made. In each run, the changes in data indicated in Table 4 were made, while all other data remained the same as in the Base Run.

Results of the Base Run and the sensitivity runs are presented in Table 5. The variables and results which are shown include:

a) The value of the objective function, Eq. (1), denoted by f.

b) The expected value of deviations from target reservoir levels at the end of the first period, $\bar{v}(\mathrm{LF}_j^1), j = 1, 2, 3$.

Table 3 Probability Distributions of Inflows.

g_1^1		g_1^2				g_2^1		g_2^2				g_3^1		g_3^2									
val.	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*	*val.*	*pr.*
3.0	0.05	8.0	0.006	10.5	0.240	5.0	0.05	11.0	0.006	13.5	0.240	2.0	0.02	3.0	0.0004	3.9	0.010	4.4	0.03	4.9	0.15	5.4	0.25
3.5	0.10	8.5	0.007	11.0	0.222	5.5	0.10	11.5	0.007	14.0	0.222	2.3	0.06	3.3	0.0012	4.0	0.006	4.5	0.01	5.0	0.0024	5.6	0.03
4.0	0.40	9.0	0.050	11.5	0.100	6.0	0.40	12.0	0.050	14.5	0.100	2.6	0.30	3.5	0.0012	4.1	0.018	4.6	0.09	5.1	0.15	5.7	0.06
4.5	0.25	9.5	0.105	12.0	0.040	6.5	0.25	12.5	0.105	15.0	0.040	2.9	0.50	3.6	0.006	4.2	0.0024	4.7	0.0072	5.2	0.036	5.9	0.06
5.0	0.20	10.0	0.230			7.0	0.20	13.0	0.230			3.2	0.12	3.8	0.0036	4.3	0.018	4.8	0.03	5.3	0.0072	6.2	0.014

Table 4 Parameter changes for sensitivity series 1 to 5 (for changes in Series 6 see text).

Series, S	*1*	*2*	*3*	*4*	*5*		
Parameter	p_{1k}	q_{1k}	r_j	S_j	$LF_j^1 = LF_j^2$		
for	$k = 1, \ldots, 14$	$k = 1, \ldots, 14$	$j = 1, \ldots, 30$	$j = 1, \ldots, 30$	$j = 1$	$j = 2$	$j = 3$
Base Run	0.2	1.0	2.0	20	10	8	6
Run S.1	1.0	0.5	0.8	3	9	7	5
Run S.2	3.0	0.1	0.3	2	8	6	4
Run S.3	5.0	0.05	0.08	1	7	5	3

c) Releases from reservoir 1 in the first period x_1^1, ($x_2^1 = x_1^1$ because they have the same cost coefficient in all runs), diversion for supply, x_4^1, and the expected value of deviation from meeting the demand, $\bar{v}(D_1^1)$.

d) Same as (c), for reservoir 2.

e) Same as (c) for reservoir 3 and for the downstream demand D_5

Note that the expected values of the deviations are not zero, which means that the optimal operation does not necessarily aim to meet all targets, not even on the average. Some average deviations are positive while others are negative. Their sign and magnitude depend on all the data of the problem, and in particular the relative magnitudes of the coefficients in the objective function and the penalties. An analysis and explanation of the results follows.

Series 1: Increasing $\mathbf{p}_{1k}$

As p_{1k} increases the penalties for positive deviations from targets decrease. This results in an increase in the expected values of most deviations from targets. At the same time the value of the objective function decreases.

Series 2: Decreasing $\mathbf{g}_{1k}$

As g_{1k} decreases the penalties for positive deviations from targets decrease. The results show the same trend as in Series 1.

Series 3: Decreasing $\mathbf{r}_j$

As r_j decreases the concavity of the objective function decreases (it is closer to linear) and it becomes more attractive to increase x_j. All flows indeed increase, except x_4 and x_9 which take water out of the system before it can generate more benefit and these are therefore decreased. The average deviations become larger, because the larger benefits from the x_j's make it attractive to derive greater immediate benefits at the risk of higher penalties later. The value of the objective function increases too.

Series 4: **Decreasing s_j**

As S_j decreases there is less operational storage in the reservoir and we must therefore expect a loss in benefit, smaller values of the decision variables and in general a tighter operation. The results indeed show this tendency.

Series 5: **Decreasing LF_j**

As the target storages in all reservoirs for both periods are reduced more water becomes available and the decision variables increase. The deviations from LF become more negative, which means that the actual volumes in storage at the end of the period deviate more in the positive direction (more water) from the target than in the Base Run.

Series 6: **Increasing the permissible probability for constraint violation**

In this series we sought to examine the effect of relaxing the probabilities in the chance constraints (in Eq. (11), for example). For the Base Run the chance constraints are not active, so we modified some of the data, as follows:

$$LI_1^1 = LF_1^1 = LF_1^2 = 5;\ LI_2^1 = LF_2^1 = LF_2^2 = 4;\ LI_3^1 = LF_3^1 = LF_3^2 = 3;$$

$p_{1k} = p_{2k} = 5$ (lower penalties for both positive and negative deviations than in the Base Run). Other data for Series 6 are:

Run	*Cost coefficient, c_j*	*Prob. in chance const., α*
6.1	8	0.05
6.2	20	0.50
6.3	20	0.90

As the probability increases, larger deviations are tolerated. As a result most average deviations increase in absolute value and the objective function increases. The increase from Run 6.1 to 6.2 is due primarily to the increase in the cost coefficients c_j from 8 to 20. The further increase in Run 6.3 is due to the larger deviations allowed.

Table 6 summarizes the output of 500 simulation runs for the optimal solution of the Base Run for period 1. Comparison of the means with the results of the optimization (last column) show good agreement. The standard deviation, minimum and maximum values generated in the simulation, and possibly the entire histograms (which are produced by SLAM) can be used to study outcomes of any selected operating policy (the optimal or any other), leading possibly to changes in prescribed input parameters, such as target reservoir levels, penalty coefficients, and probabilities for violation of chance constraints.

Table 5 Results of the Base Run and Sensitivity Runs.

Var	*Base Run*	*Series 1*			*Series 2*			*Series 3*		
		1.1	*1.2*	*1.3*	*2.1*	*2.2*	*2.3*	*3.1*	*3.2*	*3.3*
f	*412.929*	*415.112*	*418.180*	*419.146*	*416.380*	*419.823*	*420.692*	*943.716*	*1478.055*	*1746.930*
$\bar{v}(\mathrm{LF}_1^1)$	1.048	1.104	1.290	1.373	1.265	1.473	1.529	4.973	5.119	4.246
$\bar{v}(\mathrm{LF}_2^1)$	−0.276	−0.262	−0.207	−0.178	−0.220	−0.156	−0.131	3.411	3.372	2.358
$\bar{v}(\mathrm{LF}_3^1)$	−1.265	−1.279	−1.198	−1.135	−1.143	−1.051	−1.040	0.032	0.964	0.326
$X_1^1 = X_2^1$	2.636	2.665	2.758	2.779	2.745	2.849	2.877	4.599	4.672	4.236
X_4^1	2.886	2.915	3.081	3.188	3.120	3.324	3.374	3.679	1.012	0.058
$v(D_1^1)$	1.920	1.949	2.115	2.222	2.154	2.358	2.408	2.713	0.046	−0.908
$X_6^1 = X_7^1$	2.975	2.981	3.009	3.023	3.003	3.035	3.047	4.818	4.798	4.293
X_9^1	3.225	3.231	3.372	3.348	3.378	3.510	3.345	3.898	1.684	0.114
$\bar{v}(D_2^1)$	1.490	1.496	1.637	1.703	1.643	1.775	1.810	2.163	−0.051	−1.621
$X_{11}^1 = X_{12}^1$	3.319	3.330	3.336	3.338	3.323	3.337	3.340	7.040	10.000	10.000
X_{14}^1	3.096	3.149	3.182	3.190	3.117	3.186	3.201	7.040	10.000	10.000
$\bar{v}(D_4^1)$	−0.129	−0.076	−0.043	−0.035	−0.108	−0.039	−0.024	3.815	6.775	6.775
X_{15}^1	3.542	3.511	3.491	3.486	3.530	3.488	3.479	7.040	10.000	10.000
$\bar{v}(D_5^1)$	−1.423	−1.454	−1.474	−1.479	−1.435	−1.477	−1.486	2.075	5.035	5.035

Table 5 *(Continued)*

Var	*Base Run*	*Series 4*			*Series 5*			*Series 6*		
		4.1	*4.2*	*4.3*	*5.1*	*5.2*	*5.3*	*6.1*	*6.2*	*6.3*
f	*412.929*	*289.350*	*197.385*	*80.327*	*410.089*	*405.734*	*400.175*	*1084.012*	*3789.863*	*3844.554*
$\bar{v}(\mathrm{LF}_1^1)$	1.048	−0.531	−2.225	−3.225	0.332	−0.318	−1.140	2.650	3.330	3.470
$\bar{v}(\mathrm{LF}_2^1)$	−0.276	−3.225	−4.225	−5.225	−1.120	−2.100	−3.100	1.540	2.260	2.490
$\bar{v}(\mathrm{LF}_3^1)$	−1.265	−2.132	−1.934	−1.806	−2.485	−3.670	−4.758	0.305	1.240	1.630
$X_1^1 = X_2^1$	2.636	1.500	1.000	0.500	2.779	2.954	3.042	3.450	3.788	3.859
X_4^1	2.886	1.750	1.250	0.986	3.029	3.204	3.292	0.000	0.000	0.028
$v(\bar{D}_1^1)$	1.920	0.784	0.304	0.020	2.063	2.238	2.326	−0.965	−0.965	−0.937
$X_6^1 = X_7^1$	2.975	1.500	1.000	0.500	3.052	3.062	3.062	3.894	4.254	4.371
X_9^1	3.225	1.910	1.583	1.000	3.302	3.312	3.312	0.000	0.000	0.583
$\bar{v}(\bar{D}_2^1)$	1.490	0.175	−0.152	−0.050	1.567	1.577	1.577	−1.740	−1.640	−1.640
$X_{11}^1 = X_{12}^1$	3.319	1.500	1.000	0.500	3.319	3.319	3.319	8.886	10.000	10.000
X_{14}^1	3.096	1.500	1.000	0.500	3.096	3.096	3.096	8.886	9.975	9.975
$\bar{v}(\bar{D}_4^1)$	−0.129	−1.725	−2.225	−2.275	−0.129	−0.129	−0.129	5.660	6.750	6.750
X_{15}^1	3.542	1.500	1.000	0.500	3.542	3.542	3.542	8.886	10.025	10.025
$\bar{v}(\bar{D}_5^1)$	−1.423	−3.465	−3.965	−4.465	−1.423	−1.423	−1.423	4.310	5.040	5.040

Table 6 Statistics of 500 Simulation Runs with the Optimal Policy of the Base Run, and Comparison with the Analytic Solution, for Period 1.

Variable	*Target*	*Mean*	*Std. Dev.*	*Min.*	*Max.*	*Optimal value*
Objectives value		413	1.77	407	417	412.929
Penalties		15.8	1.77	11.3	21.2	
Final level, Res. 1	10	8.97	0.53	7.73	9.73	—
Deviation from LF_1^1	—	1.03	0.53	0.27	2.27	1.048
Final level, Res. 2	8	8.30	0.53	7.05	9.05	—
Deviation from LF_2^1	—	−0.30	0.53	−1.05	0.95	−0.276
Final level, Res. 3	6	7.25	0.26	6.47	7.67	—
Deviation from LF_3^1	—	−1.25	0.26	−1.67	−0.47	−1.265
Deviation from D_1^1	—	1.92	0.12	1.69	2.39	1.920
Deviation from D_2^1	—	1.49	0.31	1.02	2.22	1.490
Deviation from D_4^1	—	−0.13	0.53	−0.90	1.10	−0.129
Deviation from D_5^1	—	−1.44	0.42	−1.86	−0.46	−1.423

CONCLUSIONS

The method presented in this paper allows consideration of uncertainties in reservoir operation studies in a more general and flexible way than hitherto possible. We believe that this makes the approach attractive.

The solution algorithm is implemented on a PC/AT, using software packages which are readily available. The solution is obtained in reasonable computer time, and it always converges under the conditions stated, which are expected to hold under most cases in practice. The solutions show a reasonable and interesting response to changes in parameters introduced by the decision maker, as demonstrated in the example above.

In addition to the optimal values of the decision variables, the solution provides much useful information regarding expected future states of the system under the optimal policy. Stochastic simulation, with this policy, further enriches the information available to the decision makers for study, evaluation and modification of parameter values as he sees fit.

The FGA method by Rockafellar and Wets[9,10] is for linear constraints and a linear-quadratic objective function. Eiger[3] extended it to allow two-sided penalty functions. Wagner[16] has extended it to the case of non-linear convex constraints (global optimum guaranteed) and to the non-convex case (using a proximal point method, which leads to a local optimum).

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APPENDIX A: CONSTRAINT MATRICES FOR MULTIPERIOD MODEL

Denote by $T^1 = \{t_{kj}\}$ the matrix of coefficients in the "soft" constraints for one period, and by $\bar{T}^1$ the same matrix after we set to zero all rows belonging to constraints which do not depend on the history (e.g. those on meeting demands). Then for the two period model the matrix of the "soft" constraints is

$$T^2 = \begin{vmatrix} T^1 & 0 \\ \bar{T}^1 & T^1 \end{vmatrix} \tag{A.1}$$

For period N the corresponding matrix is composed of ($N \times N$) blocks

$$T^N = \begin{vmatrix} T^1 & 0 & 0 & & 0 \\ \bar{T}^1 & T^1 & 0 & & 0 \\ \bar{T}^1 & \bar{T}^1 & T^1 & & 0 \\ \vdots & & & \ddots & \vdots \\ \bar{T}^1 & \bar{T}^1 & \bar{T}^1 & & T^1 \end{vmatrix} \tag{A.2}$$

For the constraints which do not depend on storage we simply duplicate *the same* constraints for all periods, i.e.

$$A^N = \begin{vmatrix} A^1 & & & & \\ & A^1 & & & 0 \\ & & A^1 & & \\ 0 & & & & \\ & & & & A^1 \end{vmatrix} \tag{A.3}$$

The random variables in the constraints (A.2) are the *sums* of the inflows over periods 1 to N. The probability distributions of these variables can be obtained in either of two ways:

a) Analyzing the recorded inflows *summed* over periods 1 to N and fitting a distribution, or

b) Using the probability distributions of the inflows in each period and the dependence/independence of flows in different periods to calculate the probability distribution of their sum. For independent flows this means convolution.