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DECOMPOSITION TECHNIQUE FOR OPTIMAL DESIGN OF WATER SUPPLY NETWORKS

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A decomposition technique is suggested for optimal design of water supply networks. The general mathematical model is decomposed into two submodels which are solved iteratively. The flow variables are solved in the first submodel for a fixed value of the head variables, using a minimum concave cost flow algorithm. The head variables are solved in the second submodel for a fixed value of the flow variable using LP. The solution is usually obtained after 2 iterations, and is proven to be a local optimum. A novel form of the pump equation, based on dimensional analysis, is also presented and used as part of the optimization model.

KEY WORDS: Water supply, optimization, networks, linear programming, pumps.

INTRODUCTION

This paper presents a technique to solve the optimal capacity (i.e. pipe diameter and pump power) of a water supply network under a single load pattern. It is assumed that the layout of the network is known and includes loops. The loops are chosen to fulfill a reliability requirement, accordingly every demand node is connected to the source through two or more disjoint paths. The problem defined is non-convex, with both continuous and discrete variables (e.g. commercial pipe diameters). Many techniques have been suggested during the last 30 years; none of them is yet widely accepted.

A primary partition of the various methods can be made according to the hydraulic variables that are optimally solved in each method. The variables consist of a set of the head variables (the hydraulic heads and the head loss/gain along each hydraulic element) and a set of the flow variables. The rest of the variables are usually defined by the relationship between the head and the flow variables. Following this partition, the methods are divided into the following three categories: (1) Simultaneous solution of the optimal heads and the optimal flow variables. (2) Optimal solution of the flow discharges with respect to a fixed set of heads. (3) Optimal solution of the hydraulic heads with respect to a fixed set of flows.

The methods in the first group are usually based on non-linear programming (NLP) or enumeration techniques. The works of Watanatada[1], Shamir[2] and, El-Bahrawy and Smith[3] make use of the Quasi-Newton methods, penalty functions, and the General-Reduced-Gradient (GRG). These techniques are relatively complicated and are subject to convergence difficulties, especially at low values of the discharges. Pitchai[4] and Gessler[5] used enumeration techniques by which the possible diameters for each pipe are searched in order to find the best set of diameters.

The methods in the second group solve the optimal flow discharges for a fixed set of heads. Knowing the values of the heads, the problem is reduced to a concave problem with a separable objective function and linear constraints. Methods that belong to this group were proposed by Lai and Schaake[6], Quindry *et al.*[7] and Rowell and Barnes[8]. An iterative procedure is used by Quindry *et al.*[7] by which the fixed head distribution is improved after each iteration. The reduced problem may be solved by means of separable programming (SP) which converges to a local optimum.

The methods in the third group solve the optimal head distribution for a fixed set of flows. Knowing the flows, the problem is reduced to a convex problem, with a separable convex objective function and linear constraints. The reduced problem results in a global optimum. Furthermore, the convex problem may be formulated as a linear problem using a variable transformation, i.e., length of pipe segments of different commercial diameters instead of the head loss/gain variables. Many methods were proposed for utilizing these attractive features. Among them are the works of Cembrowicz and Harrington[9], Alperovits and Shamir[10], Bhave[11], Morgan and Goulter[12], Fujiwara *et al.*[13] and Kessler and Shamir[14].

The methods of the third group are considered to be efficient and reliable due to their relatively simple procedures. On the other hand, the methods in the second group has the advantage of better initial solutions. That is, using a linear surface of heads or a fixed headloss in each pipe provides a better starting point compared with an arbitrary initial flow distribution, as is needed for methods of the third group.

A common feature for all the three groups (except for the enumeration techniques) is that the optimal solution is obtained through a search procedure. The rate of convergence and the stability depend heavily on the step direction and size made in each iteration. In this work a different approach, utilizing the advantages of the second and third groups, is examined. The general problem is decomposed into convex and concave sub-problems which are solved iteratively. The convex problem is solved by LP and the concave problem is solved by the minimum concave cost flow algorithm. Convergence is usually achieved after 2 iterations.

The decomposition method was first suggested by Morgan and Goulter[15] to determine the layout and pipe diameters while maintaining some degree of reliability. A slightly modified version was later introduced by Goulter and Morgan[16] to ensure the presence of two disjoint paths between the source and every demand node. In our work the original version of Morgan and Goulter[15] is reformulated, using new sets of variables and graph theory techniques. The optimal solution is analytically proved to approach a spanning tree configuration. The model is extended to handle reservoirs and pump stations, where a variable speed pump is directly represented as a function of its impeller diameter and is restricted to operate at its most efficient point.

THE MATHEMATICAL MODEL

A water supply network is an ordered set of hydraulic elements by which water is transferred from sources to a distributed set of consumers (demand nodes). The hydraulic elements are pipes, pumps and valves, which are all two terminal elements. Each hydraulic element has an equation which describes the relationship between the discharge and the headloss, as function of the element capacity. It is the purpose of the optimal design to determine the minimum cost capacity of the network by which the water is properly provided to the consumers.

Assuming a flow direction for each hydraulic element, the network is described by a directed graph, $G(N, E)$, where N is the set of (nn) nodes and E is the set of (ne) directed edges (elements). Edges for which the flow direction is not known in advance may be replaced by two parallel edges, one in each direction. For a multi-source network an artificial node is added, named the root node, connected by a set of artificial edges, one to each source. The root node becomes the only source in the new network, for which a fixed head is assumed. Hereafter the network topology is represented by the graph incidence matrix, $\bar{\mathbf{R}}$ (size $nn \times ne$), whose elements are

$$\bar{R}_{ij} = \begin{cases} +1 & \text{if arc } j \text{ is directly away from node } i \\ -1 & \text{if arc } j \text{ is directed toward node } i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Each column of $\bar{\mathbf{R}}$ corresponds to an edge and has exactly two non-zero entries, one $(+1)$ and the other (-1). Each row in $\bar{\mathbf{R}}$ corresponds to a node, where the non-zero entries indicate the edges incident to that node. Since the rank of $\bar{\mathbf{R}}$ is $(nn - 1)$, it is possible to eliminate one row without loss of information (Deo[17], p. 139). Eliminating the row of the source node, R_s, the reduced incidence matrix, $\mathbf{R}$, is implicitly defined by:

$$\bar{\mathbf{R}} = \left(\frac{R_s}{\mathbf{R}}\right) \tag{2}$$

The network is subjected to two types of hydraulic constraints. The first states that the sum of inflows and outflows at each node is equal to zero. This constraint, called the flow constraint, is given by the set of linear equations:

$$\mathbf{Rq} = -\mathbf{w} \tag{3}$$

where $\mathbf{q}$ (size ne) is the vector of flows in the edges and $\mathbf{w}$ (size $nn - 1$) is the outflow from each node (negative for inflow).

The other hydraulic constraint states that the head loss/gain around each loop is zero, and that the head at every node is confined to be within a given range of values. This constraint, called the head constraint, is defined on the set of loops and the set of paths between the source and every node. An alternative formulation which removes the need to select paths and loops, is based on the following consideration: the headloss along each edge is equal to the hydraulic head difference between its two end-nodes. Thus the head constraints are expressed by:

$$\bar{\mathbf{R}}^T\mathbf{h} - \Delta\mathbf{h} = \mathbf{0} \tag{4}$$

$$\mathbf{h}_{\min} \leq \mathbf{h} \leq \mathbf{h}_{\max} \tag{5}$$

where $\mathbf{h}$ (size nn) is the vector of the hydraulic heads and $\Delta\mathbf{h}$ (size ne) is the vector of the headlosses. Define a new set of variables, $\mathbf{h}^+ = \mathbf{h} - \mathbf{h}_{\min}$, to be the surplus head with respect to the minimum required head at each node. It can be shown (Kessler and Shamir[14]) that the last two constraints can be converted into:

$$\mathbf{R}^T\mathbf{h}^+ - \Delta\mathbf{h} = \mathbf{R}^T\Delta\mathbf{h}_p \tag{6}$$

$$\mathbf{0} \leq \mathbf{h}^+ \leq (\mathbf{h}_{\max} - \mathbf{h}_{\min}) \tag{7}$$

where $\Delta\mathbf{h}_p$ is the maximum headloss allowed along the path from the source to each demand node.

The relationship between the two types of variables, the head and the flow, is given for each hydraulic element by its element equation. For a pipe, the Hazen–Williams equation is commonly used:

$$\Delta h_e = k_e(q_e/chw_e)^{1.852}d_e^{-4.87}a_e \tag{8}$$

where chw_e, d_e and a_e are the smoothness coefficient, the diameter and the length of pipe e, respectively. k_e is a constant which depends on the units used.

The pump equation describes the relationship between the discharge, q, and the head gain, Δh, as function of the pump's physical parameters. Experimental tests show that the significant parameters are: pump size, represented by the impeller diameter (b), its angular velocity (n), the kinematic viscosity (ν), the added pressure, represented by $g\Delta h$, and the discharge (q). These parameters are based on two physical variables, namely length and time. Following the Buckingham theorem the pump operation is governed by (5 parameters minus 2 physical variables =) 3 dimensionless parameters. The dimensionless parameters may appear in various forms, corresponding to the different combinations of choosing 3 parameters out of 5.

Of the three dimensionless parameters, the Reynolds number is of negligible influence under typical operating conditions. The two remaining dimensionless parameters are usually selected to be the discharge coefficient, q/nb^3, and the head coefficient, $g\Delta h/n^2b^2$. For a series of homologous (similar geometry) pumps of various sizes, these parameters define a single dimensionless curve as shown in Figure 1. The most efficient working point along this curve, (ϕ_q, ϕ_h), defines the most efficient working points for all the pumps within that series (it depends on q/nb^3 alone). For the purpose of optimal design we consider only the case where a pump is working at its most efficient point.

An alternative pair of dimensionless parameters, which is more useful for our purposes, define the most efficient points according to:

$$\phi_b = g\Delta h q^{-2}b^4 \tag{9}$$

$$\phi_n = (g\Delta h)^{-3/4}q^{1/2}n \tag{10}$$

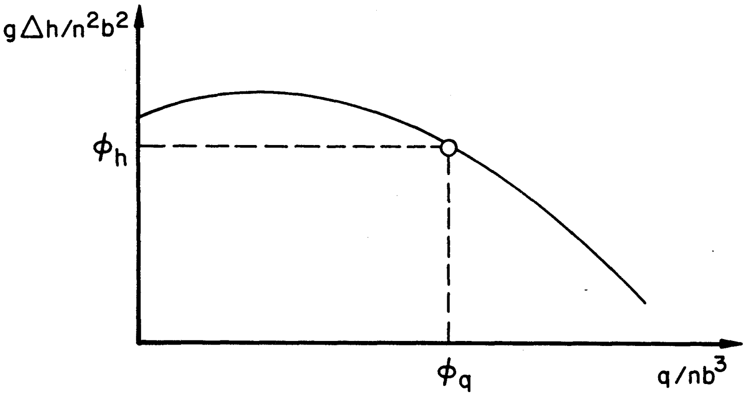

Figure 1 Typical dimensionless pump curves.

where ϕ_b and ϕ_n are called the impeller coefficient and the specific velocity, respectively. Again, these parameters are of constant values for a given type of pump series, i.e. centrifugal, mixed flow or propeller pump.

The type of pump can be determined prior to the optimal solution, based on the observations that each type is best suited for a particular range of q and Δh. Centrifugal pumps have a better efficiency at low discharge and high pressure range, while propeller pumps are more efficient at high discharges and low pressure range (Davis and Sorensen[18]). Given the type of pump (or the ϕ_b and ϕ_n values), the discharge-head relationship at the most efficient point is derived from Eqs. (9) and (10) as follows:

$$\Delta h = (\phi_b g^{-1}) b^{-4} q^2 \tag{11}$$

$$\Delta h = (\phi_n^{-4/3} g) n^{4/3} q^{2/3} \tag{12}$$

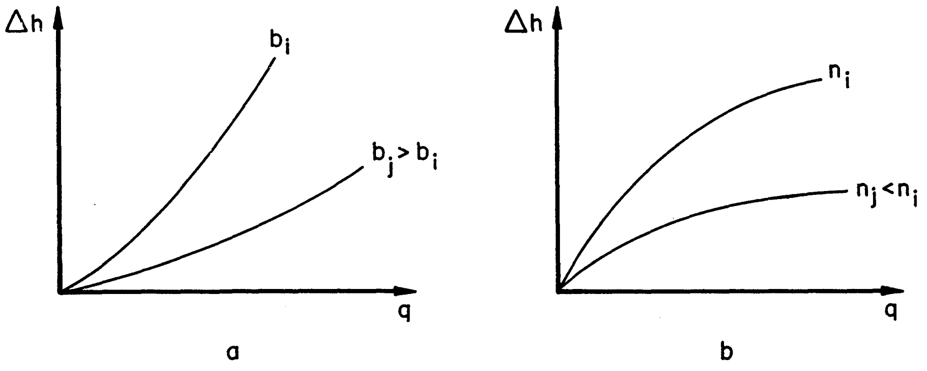

Figure 2 Pump curves at most efficient conditions for:
a) different sizes of the impeller
b) different angular speeds.

Equation (11) defines a curve $(q, \Delta h)$ for every diameter, b, while Eq. (12) defines a curve for every angular velocity, n. The two sets of curves are schematically shown in Figure 2 such that for every pair $(q, \Delta h)$ there corresponds a unique pair (b, n). Having a series of variable speed pumps, the only design variable is the impeller diameter, b, while the angular velocity, n, is adjusted to run the pump at its most efficient point. The pump equation is therefore fully described by Eq. (11), restricted to a given series of variable speed pumps.

The goal of the optimal design is to minimize the total cost of the system. That is, the cost of the pipelines, the pumps and the reservoirs. The cost of a pipe can be approximated (Quindry *et al.*[7]) by $c_e d_e^\lambda a_e$, where c_e and λ are obtained by regression and a_e is the length. If we assume that the pipe material (chw_e) is fixed, and substitute d_e from Eq. (8) into the above expression, the pipe cost is given by:

$$\text{cost}(\text{pipe}_e) = CD_e q_e^\alpha \Delta_e^{-\beta} \tag{13}$$

where, $CD_e = c_e k_e^{\lambda/4.87} chw^{-1.852\lambda/4.87} a_e^{1+\lambda/4.87}$ is a constant value for each pipe, with $\alpha = 1.852\lambda/4.87$ and $\beta = \lambda/4.87$. For $1 \le \lambda \le 2$, as is found for usual cost data, $0 \le \beta \le \alpha \le 1$. This means that Eq. (13) is concave in q and convex in Δh.

The cost of a pump should include the capital cost of the pump and the costs of energy and maintenance. The cost of the energy and maintenance can be approximated as linearly proportional to pump power. Capital cost of the pump depends on the pump type, the make and market conditions. Still, it can be approximated by a function of the pump size according to $c_e b_e^\eta$. Substituting the pump Eq. (11) into this cost function, the total cost of a variable speed pump is given by:

$$\text{cost}(\text{pump}_e) = CP_e q_e^\gamma \Delta h_e^{-\delta} + CHP_e q_e \Delta h_e \tag{14}$$

where $CP_e = c_e(\phi_b g^{-1})^{\eta/4}$ is a constant, $\gamma = \eta/2$ and $\delta = \eta/4$. For $1 \le \eta \le 2$, as is usually found, $0 \le \delta \le \gamma \le 1$ which implies that Eq. (14) is concave in q and convex in Δh. CHP_e is the cost of the energy required to pump 1 m^3/hr through a pressure difference of $1m$, over the entire life-span of the project, in present value terms, assuming the efficiency is known.

The cost of the reservoirs is considered here only with respect to their elevation above ground level (Alperovits and Shamir[10]). Assuming a constant marginal cost, CR_r, for raising the reservoir by one meter, the cost is given by: $CR_r h_r^+$, where h_r^+ is the average water level above the ground.

Incorporating all of the above, the optimal design model is:

$$\mathbf{GP}:\text{Min}\left\{\sum_{e\in P} CD_e q_e^\alpha \Delta h_e^{-\beta} + \sum_{e\in S}(CP_e q_e^\gamma \Delta h_e^{-\delta} + CHP_e q_e \Delta h_e) + \sum_{r\in \mathrm{RES}} CR_r h_r^+\right\} \tag{15}$$

Subject to:

$$\mathbf{R}\mathbf{q} = -\mathbf{w} \tag{16}$$

$$\mathbf{R}^T\mathbf{h}^+ - \mathbf{\Delta h} = \mathbf{R}^T\mathbf{\Delta h}_p \tag{17}$$

$$\mathbf{0} \le \mathbf{h}^+ \le (\mathbf{h}_{\max} - \mathbf{h}_{\min}) \tag{18}$$

$$\mathbf{q} \ge \mathbf{q}_{\min} \tag{19}$$

$$\mathbf{\Delta h} \ge \mathbf{0} \tag{20}$$

the sets P and S correspond to the set of the pipelines and pump stations, respectively such that $E = P + S$. The set RES is the set of all the reservoirs in the network which is subset of the node set N. Due to the presence of pumps, the incidence matrix $\mathbf{R}$ in Eq. (17) is determined according to the headloss sign rather than the flow direction. That is $R_{i,j}$ is $(+1)$ if the head at node i is greater than the head of the node on the other side of edge j. Constraint (19) does not allow a tree-shaped solution, as would result if $q_{\min} = 0$. It is needed to maintain the initial layout by which at least two disjoint paths (of distinct edges) lead from the source (or sources) to each demand node, thus providing some measure of reliability. A comprehensive consideration of reliability depends on more than this simple requirement, but goes beyond the scope of our analysis here.

THE DECOMPOSITION TECHNIQUE

The model GP consists of two types of decision variables: the head variables, $\mathbf{h}^+$ and $\Delta \boldsymbol{h}$, and the flow variables $\mathbf{q}$. By observing the constraint set (Eqs. (16) and (17)) it is seen that the two types of variables are independent of each other; that is, a change in one variable type will not affect the feasibility of the other variable type. This makes it possible to solve optimally the head variables for a fixed value of the flow variables, and *vice versa.*

Another significant property of GP is the unique structure of its objective function. For each hydraulic element (pipe or pump) it contains an expression given as a function of the head and flow in that element. Fixing one type of variable, the objective function becomes a separable function of the other type. This function is convex in $\Delta\mathbf{h}$ for a fixed value of $\mathbf{q}$, and concave in $\mathbf{q}$ for a fixed value of $\Delta\mathbf{h}$. Moreover, the terms included in the convex function are all monotone decreasing while the terms included in the concave function are all monotone increasing.

Looking for the best technique to solve GP, it is suggested to utilize the above two properties to full advantage. The independence of the variable sets makes it attractive to decompose GP into two subproblems. The first solves for the flows with a given value of the heads. The second solves for the heads with a given value of the flows. The second property, i.e., the special structure form of the objective function, discussed above, enables an efficient technique to be applied to each submodel. The minimum concave cost flow algorithm is found most suitable to solve for the flows, while an LP technique is used to solve for the heads.

The fixed head model

Assuming a fixed feasible value for the head variable $\mathbf{h}^+$ and $\Delta\mathbf{h}$ (i.e., they satisfy constraint (17)), the general model GP is reduced to the following FH sub-model:

$$\mathbf{FH}\!:\left\{\mathrm{Min} \sum_{e\in P} K1_e q_e^{\alpha} + \sum_{e\in S} (K2_e q^{\gamma} + K3_e q_e)\right\} \qquad (0 \le \alpha, \gamma \le 1) \tag{21}$$

Subject to:

$$\mathbf{Rq} = -\mathbf{w} \tag{22}$$

$$\mathbf{q} \geq \mathbf{q}_{\min} \tag{23}$$

where, for pipes, $K1_e = CD_e \Delta h_e^{-\beta}$, and for pumps $K2_e = CP_e \Delta h_e^{-\delta}$ and $K3_e = CHP_e \Delta h_e$. The essential difference between this model and previous fixed head models (Lai and Schaake[6], Quindry *et al.*[7], Morgan and Goulter[15]) is due to the decision variables. While FH solves directly for the flow in each edge, the previous models solve for the pipe diameters using a variable transformation technique. Such a transformation changes the structure of the continuity constraints, thus making it impossible to use the efficient minimum cost flow algorithms. The use of the minimum cost flow algorithm is described later in this chapter.

FH is a minimum concave cost flow problem for which more than one local optimum may exist. Because there are only linear constraints, it can be proved (Hadley[19], p. 91) that every local optimum corresponds to one of the extreme points defined by the polyhedron. Since the constraint set of FH is defined by the incidence matrix **R**, each extreme point is a basis of **R**. A well known result in graph theory (Deo[17], p. 141) is that every basis of **R** defines a unique spanning tree of the network. It follows that the solution of FH is always a spanning tree, consisting of all edges whose flows are above the minimum value. In the remaining edges the flow is at its minimum.

Solving FH is a difficult task due to its multiple local optima. The global optimum may be found only by a search over all the local optima (i.e., the spanning trees). Such a search presents a formidable computational problem which has been identified as NP-complete (Jensen and Barnes[20]). That is, no algorithm has been found to solve any problem in this class which is bounded in polynomial computation time. Other techniques such as non-linear programming (NLP), separable programming (SP) or successive linearization techniques converge to a local optimum in the vicinity of the starting point. Using such techniques it is often found (Minoux[21]) that the improvement upon a given starting point is much smaller than the difference (in cost) between the starting point and the global optimum. As a consequence, the resulting local optimum will be good only if the starting point was already a good approximation to the minimum cost solution.

A good starting point might be provided by the shortest-path spanning tree as suggested by Bhave[11]. The shortest-path spanning tree is defined as a spanning tree which consists of all the shortest paths between the source (root) and every demand node. Evaluating the validity of such an assumption is made possible by examining the Kuhn–Tucker points of FH. It can be proved (Yaged[22]) that the optimal solution of the minimum concave cost flow problem is the shortest-path spanning tree τ where the 'length' of each edge is equal to $d(\text{cost}_e)/dq_e$ for all $q_e \in \tau$. That is, the length is taken as the cost that would result from a unit increase of the flow, q_e, through edge e. Hereafter, we shall use the term 'weight', instead of 'length', to distinguish it from the physical length of an edge.

Using the general cost function, $k_e q^{\alpha}{}_e \Delta h_e^{-\beta}$, for an hydraulic element (pipe or pump), the weight of that element is equal to:

$$\text{weight}_e = d(\text{cost}_e)/dq_e = (\alpha k_e \Delta h_e^{-\beta}) q_e^{\alpha-1} \tag{24}$$

An immediate result of Eq. (24) is that the weight of each element is a function of its flow. However, the flow is unknown and therefore the shortest-path spanning tree cannot be determined unless $\alpha = 1$, i.e., unless the pipe cost is a linear function of its flow. To find a good starting point the weights may be approximated by linearizing the element cost function. The shortest-path spanning tree is later solved by Dijkstra's algorithm (Jensen and Barnes[20]), thus defining a spanning tree in the vicinity of the optimal one.

The linearization is made, as shown in Figure 3, along the same range for all the hydraulic elements. Attempts to improve the linear approximation by using different flow ranges for different hydraulic elements (Quindry *et al.*[7] and Goulter and Morgan[16]) are not recommended here due to the following rationalization. Allocating a high flow range for a specific element leads to a weight reduction of the element's weight, as requested by the concave nature of Eq. (24). Consequently, this element is more likely to be included in the shortest spanning tree, compared with an element of a lower flow range. It follows that an early allocation of flow ranges determines the final solution to a large extent.

Having a starting point, FH can be solved (to the nearest local minimum) by one of the general optimization techniques such as NLP, SP or successive LP solutions. The last technique was used by Lai and Schaake[6], Quindry *et al.*[7] and Morgan and Goulter[15]. More efficient techniques are those of minimum cost flow algorithms which completely eliminate the need for carrying and updating the LP basis inverse. Such techniques require less computer storage and produce solutions in about one-hundredth of the computation time required by the general techniques (Kennington and Helgason[23]).

We adopt a primal algorithm of minimum cost flow, as suggested by Jensen and Barnes[20] (p. 175). Having a feasible solution, the marginal cost of each edge,

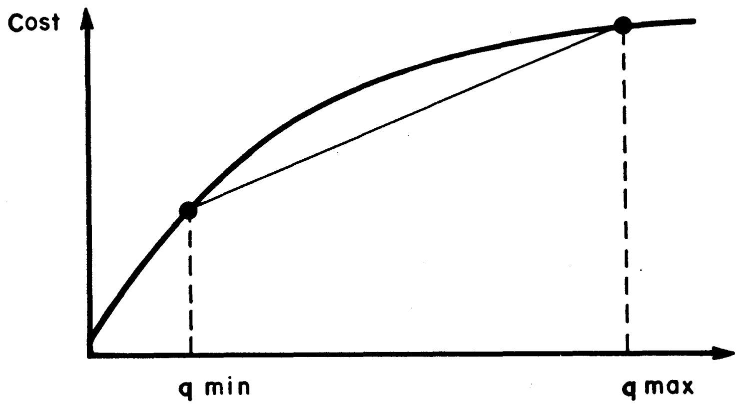

Figure 3 Linear approximation of the pipe cost function in the FH submodel.

$d(\text{cost})_e/dq_e$, is approximated and a negative cycle is detected by Dijkstra's algorithm. The flow around the negative cycle is increased until an edge becomes of zero flow (saturated). The process terminates when no negative cycles remain. The algorithm was originally proposed for minimum cost flow problems having linear cost functions. It is due to the non-decreasing property of the cost function that such an algorithm can be easily updated for a concave flow problem. Increasing the flow around a negative cycle, the concave cost function is decreased monotonically (even accelerated) until an edge becomes saturated. The only difference between the linear and the concave algorithm is that the marginal cost of a concave function should be updated after each iteration while the marginal cost of a linear function is constant.

The fixed flow model

Assuming a fixed value for the flow variables, $\mathbf{q}$, which are feasible (i.e., satisfy the node continuity constraints (16)), the general model GP is reduced to the following FQ sub-model:

$$\mathbf{FG}\text{:Min}\left\{\sum_{e\in P} K4_e \Delta h_e^{-\beta} + \sum_{e\in S} (K5_e \Delta h^{-\delta} + K6_e \Delta h_e) + \sum_{r\in RES} K7_r h_r^{+}\right\} \qquad (0 \le \beta, \delta \le 1) \tag{25}$$

Subject to:

$$\mathbf{R}^T\mathbf{h}^+ - \Delta\mathbf{h} = \mathbf{R}^T\Delta\mathbf{h}_p \tag{26}$$

$$\mathbf{0} \le \mathbf{h}^+ \le (\mathbf{h}_{\max} - \mathbf{h}_{\min}) \tag{27}$$

$$\Delta\mathbf{h} \ge \mathbf{0} \tag{28}$$

where $K4_e = CD_e q_e^{\alpha}$ for pipes, $K5_e = CP_e q_e^{\gamma}$ and $K6_e = CHP_e q_e$ for pumps, and $K7_r = CR_r$ for reservoirs. The model consists of a convex and separable objective function and a linear set of constraints. Using a piecewise linear approximation of the objective function, an LP procedure may be directly used (Kennington and Helgason[23], p. 184).

In order to get pipe diameters as discrete commercial values, a variable conversion is applied, originally suggested by Karmeli *et al.*[23] and later by Alperovits and Shamir[10]. The new variable $x_{e,d}$ is the segment length in pipe (e) having a diameter $d \in D_e$. D_e is the set of commercial pipes of different diameters which are available for pipe (e). The transformation into the new variables is given by:

$$\sum_{d\in D_e} j_{e,d} x_{e,d} = \Delta h_e \tag{29}$$

$$\sum_{d\in D_e} x_{e,d} = a_e \tag{30}$$

where $j_{e,d}$ is the hydraulic gradient in pipe segment (e) with diameter (d) for the known flow q_e, and a_e is the pipe length. The cost function associated with each pipe is given in a linear form:

$$\text{cost(pipe)} = \sum_{d \in D_e} c_{e,d} x_{e,d} \tag{31}$$

where $c_{e,d}$ is the cost per unit of length the segment (e) with a diameter (d).

Substituting Eqs. (29) to (31) into FQ and applying a piecewise linear approximation for the pump cost, the model is linearized and may be solved by a conventional LP technique. A detailed description of the linearized model is given in Kessler[25] and Kessler and Shamir[14]. FQ is basically similar (except for the pump element) to other fixed flow models such as Alperovits and Shamir[10] and Morgan and Goulter[15]. In the previous models the head constraints are defined according to a selected set of loops and paths. In this model the head constraints are uniquely defined by the incidence matrix $\mathbf{R}$.

The algorithm

Models FH and FQ are solved sequentially such that the output of the one sub-model is used as an input to the second and *vice versa*. The process is terminated when no further improvement is indicated. Following is the complete algorithm:

1) Assume a reasonable head distribution over the network (an hydraulic head for each node).

2) Solve FH for $\mathbf{q}$ using the head distribution found in the previous step. For the first iteration the starting point is the shortest path spanning tree. For later iterations the starting point is equal to the last solution of FH.

3) Compare the last two FH solutions. If they are identical then stop.

4) Solve FQ for $\mathbf{h}^+$ and $\Delta\mathbf{h}$, using the flow distribution as obtained by FH. Go to step 2.

This algorithm performs an alternating search in the $\mathbf{q}$ and $\Delta\mathbf{h}$ directions. Since $\mathbf{q}$ and $\Delta\mathbf{h}$ are independent variables, the above procedure is equivalent to the Coordinate Descent Method (Luenberger[26]) for which global convergence is proved. The algorithm may also be interpreted as a search within a family of the spanning trees, as defined by all edges with $q_e > q_{\min}$. Each time FH is solved a new spanning tree is identified, better than the previous one. The algorithm is terminated when the same spanning tree appears in two sequential iterations. Evidently, since the solution of FH is always a spanning tree, the global optimum of GP must also be a spanning tree (corresponding to the optimal head distribution). This confirms the results of many previous works which have claimed that the optimal solution of a water supply network (under a single load pattern) tends to eliminate all pipes that do not belong to a spanning tree.

The convergence rate of the above algorithm is independent of the step size used in other search methods. In fact, in all the tests we have carried out the solution was obtained after 2 iterations. Such a result may be explained, although not proved, by the following reasoning: Starting from an assumed head distribution and solving FH, an optimal flow distribution is obtained at one of the extreme points of the polyhedron. Next FQ is solved and a new iteration begins. The difference between the first FH and the second FH appears only in the coefficients of the cost function. The polyhedron remains unchanged. Those coefficients were changed via a change in the head variables by FQ, in order to optimally match the first flow distribution. It follows the first flow distribution is more likely to become the solution of the next iteration and therefore to become the optimal one.

EXAMPLES

Optimal solutions of two simple networks are presented in this section. The first network is of gravitational flow while the second network includes a pump and a reservoir. Both are based on the network originally solved by Alperovits and Shamir[10]. Other works which solve the same network are of Quindry *et al.*[7], Goulter *et al.*[27], Fujiwara *et al.*[13] and Kessler and Shamir[14]. In order to compare our results and those in previous works, the iterative procedure starts from an assumed flow distribution instead of a head distribution.

Example No. 1

The first network consists of two loops and a single source at a $+210$ m fixed head node, as shown in Figure 4. The ground elevation and the demands are given in

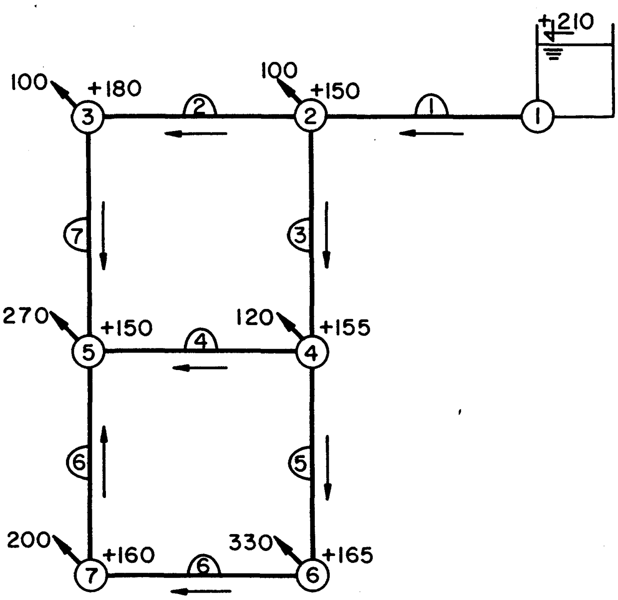

Figure 4 Simple gravitational network (example 1).

Table 1 Pipe cost per 1 meter (from Alperovits and Shamir[10])

Dia. (*inches*)	1	2	3	4	6	8	10	12	14	16	18	20	22	24
Cost (*units*)	2	5	8	11	16	23	32	50	60	90	130	170	300	550

Figure 4 along with the assumed flow directions. The pipelines are all 1000 m with smoothness coefficient $chw = 130$. A minimum head of 30 m above the ground level is required at each demand node and a minimum discharge of 10 m^3/hr is required in each pipe. Table 1 presents the total cost per one meter of pipe length for different commercial pipe diameters.

First, FQ is solved with an initial flow distribution shown in Table 2 as used in Alperovits and Shamir[10]. The model is linearized by means of the variable transformation (Eqs. (29) to (31)) where 5 different commercial diameters are available for each edge. The resulting headlosses are given in Table 2 under the subtitle FQ(1) with the associated total cost of 474,900 units.

Next, FH is solved with the resulting headlosses of FQ. Prior to the solution, the objective function of FH is determined according to the following steps:

- An exponential curve is fitted to the pipe cost data as shown in Figure 5. The pipe cost (units) as function of its diameter (inches) is given by:

$$\text{cost(pipe)}_e = 0.184 d_e^{2.23} \tag{34}$$

- The pipe diameter as function of the discharge (m^3/hr), the headloss (m) and the pipe length (km) is expressed by Hazen–Williams equation:

$$d_e = 11.755(q_e/chw_e)^{0.380}(\Delta h_e/a_e)^{-0.205} \tag{35}$$

- Eq. (35) is substituted into Eq. (34) to get:

$$\text{Cost(pipe)}_e = 44.8 chw_e^{-0.847} a_e^{1.457} \Delta h_e^{-0.457} q_e^{0.847} \tag{36}$$

- For $a_e = 1$ km and $chw = 130$ the cost function of pipe e is:

$$\text{Cost(pipe)}_e = 0.7257 \Delta h_e^{-0.457} q_e^{0.847} \tag{37}$$

where Δh_e is the resulting headloss of FQ.

Table 2 Summary of the results of example 1.

Element No.	*Initial q (m^3/hr)*	*FQ(1)* $\Delta h(m)$	*FH(1)* $q(m^3/hr)$	*FQ(2)* $\Delta h(m)$	*FH(2)* $q(m^3/hr)$	*Optimal diameters and pipe lengths (m)*	
1	1120	4.2	1120	6.8	1120	1000(18″)	
2	220	15.8	350	13.2	350	934(10″)	66(12″)
3	800	6.4	670	4.6	670	1000(16″)	
4	30	19.4	10	18.6	10	287(2″)	713(3″)
5	650	4.4	540	3.6	540	164(14″)	836(16″)
6	320	5.0	210	5.0	210	891(10″)	109(12″)
7	120	10.0	250	10.0	250	181(8″)	819(10″)
8	120	10.0	10	10.0	10	80(2″)	920(3″)

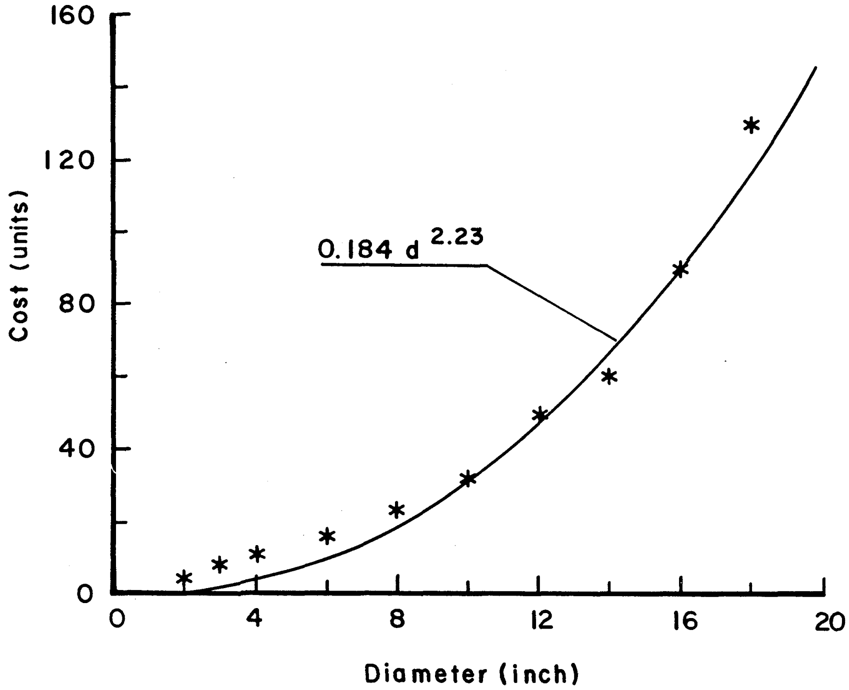

Figure 5 Best fitted curve of the pipe cost data in Table 1.

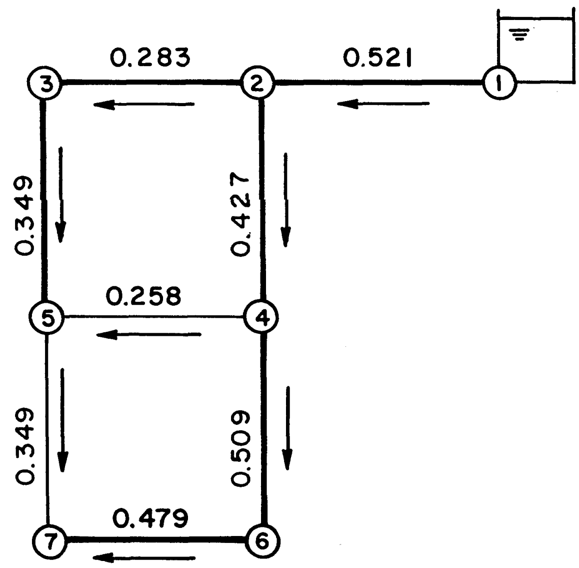

Figure 6 Shortest-path spanning tree in example 1 which is the starting point in FH.

The starting point used for FH is defined by the shortest-path spanning tree with weights equal to:

$$d(\text{cost}_e)/dq_e = 0.615\Delta h_e^{-0.457} q_e^{-0.153} \quad (38)$$

Since q_e is unknown, the weights are estimated by the linearization of the cost function with respect to q_e. Assuming that the linearization is made on the same flow range for all pipes, there is no need to actually calculate the best fitted linear coefficient. It is only the ratio between the pipe weights that is important in the shortest-path tree determination. Since all the pipes are of the same length and smoothness, their weights are proportional to $\Delta h_e^{-0.457}$. The weights and the resulting shortest-path tree are shown in Figure 6. Applying the minimum cost flow algorithm, no negative cycle is detected and therefore the starting point is also the final point, i.e., a local optimum. The resulting flow distribution is defined by a spanning tree (the same as in Figure 6), with minimum flow of 10 m^3/hr at edges outside the spanning tree. A list of the discharges is given in Table 2 under the subtitle FH(1).

FQ and FH are solved again thus defining new head and flow distributions (Table 2 under FQ(2) and FH(2), respectively). The two flow distributions, FH(1) and FH(2), are identical and therefore a local optimum is achieved. The final cost is 417,500 units which is equal to the cost recently obtained by Kessler and Shamir[14]. Previous works of Alperovits and Shamir[10], Quindry *et al.*[7] and Fujiwara *et al.*[13] yielded values of 497,525, 441,552 and 415,500 units, respectively. Although the solution of Fujiwara *et al.*[13] has a lower cost than the present solution, it consists of a minimum flow of 1.01 m^3/hr in edge 8. Relaxing the minimum flow constraint (i.e. $q \geq 0$ instead of $q \geq 10$) the minimal cost solution becomes 400,155 units.

Example No. 2

The second network is an extension of the first, in which a pump with variable speed and a reservoir are added. The additional variables are the diameter of the pump impeller and the average water level in the reservoir. The network is shown in Figure 7 along with the assumed flow directions, the demands and the ground elevation at each node. The pipe lengths and the smoothness coefficients are the same as in example 1.

The pump is represented by an hydraulic element of zero length between nodes 1 and 11. Based on daily operational considerations, a constant discharge of 420 m^3/hr is pumped into the network. An additional discharge of 800 m^3/hr is supplied by a reservoir which is placed 100 m away from node 7. Because of the presence of the pump, the pressure head at each demand node was limited not to exceed 70 m.

Equation (11) is used to describe the pump equation with a coefficient, ϕ_b, that was adjusted according to a series of commercial pumps. The fitted pump equation is:

$$\Delta h = 3 \times 10^{-6} q^2 b^{-4} \quad (39)$$

where Δh is the gain head (m), q is the discharge (m^3/hr) and b is the diameter of the impeller (m). The cost of the pump station is:

$$\text{cost(pump + energy)} = 150b^{1.3} + 2.0q\Delta h \text{ (units)} \quad (40)$$

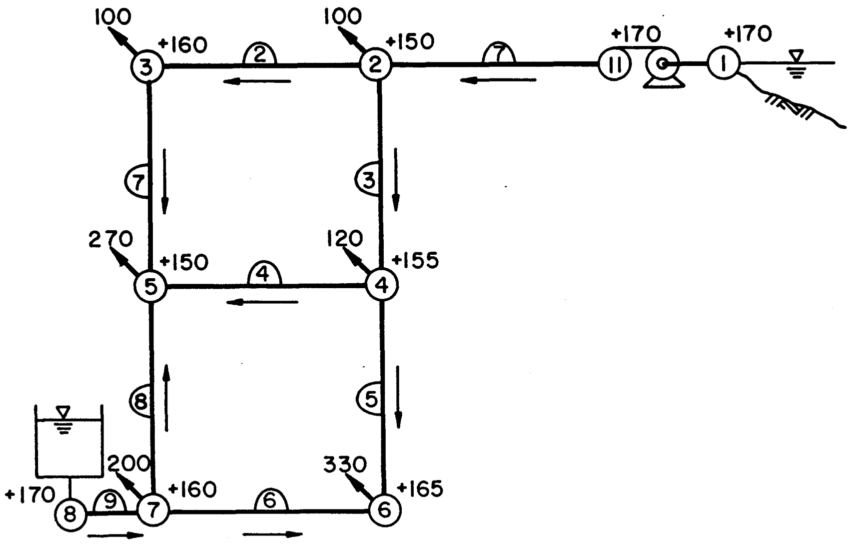

Figure 7 Simple network with pump and reservoir (example 2).

substituting (39) into (40), the total cost of the pump station is:

$$\text{Cost(pump + energy)} = 950q^{0.65}\Delta h^{-0.325} + 2.0q\Delta h \text{ (units)} \qquad (41)$$

A marginal cost for raising the average water level in the reservoir (i.e., lifting the reservoir), CR_r, is approximated as 2000 units/m.

Starting from an initial flow distribution, shown in Table 3, FQ is solved following a linear piecewise approximation of the pump cost function (Eq. (41)). The linearization is made only with respect to Δh only since q is constant, equal to 420 m^3/hr.

Table 3 Summary of the results of example 2.

Element No.	*Initial q (m^3/hr)*	*FQ(1)*	*FH(1)*	*FQ(2)*	*FH(2)*	*Optimal diameters and pipe lengths (m)*
		$\Delta h(m)$	*$q(m^3/hr)$*	*$\Delta h(m)$*	*$q(m^3/hr)$*	
1	420	19.3	420	19.3	420	1000(10″)
2	160	27.2	110	19.4	110	1000(6″)
3	160	9.6	210	14.2	210	847(8″) 153(10″)
4	10	27.6	10	25.8	10	460(2″) 540(3″)
5	30	12.6	80	10.8	80	1000(6″)
6	300	10.3	250	7.4	250	1000(10″)
7	60	10.0	10	20.6	10	335(2″) 665(3″)
8	200	25.3	250	22.4	250	8(6″) 992(8″)
9	700	1.0	700	0.9	700	100(14″)
Pump		−66.5	420	−69.3	420	
Reservoir level		41.3	38.3	38.3	38.3	

Table 4 Piecewise linear approximation of the pump cost function cost(pump) $= \mu CPq^{\gamma}\Delta h + CHPq$

Interval	$\Delta h(m)$	μ
1	0 to 6.4	−0.0625
2	6.4 to 18.6	−0.0114
3	18.6 to ∞	−0.0028

The pump head interval and their marginal cost (slopes) are shown in Table 4. The resulting headlosses are shown in Table 3 under FQ(1). The network cost at this stage is 316,850 units with an average water level at the reservoir of 41.3 m above ground level.

Next, FH is solved using the last head distribution as input. A starting point is determined similar to that in the first example. As a result, a new flow distribution is obtained (FH(1) in Table 3). This flow distribution defines a spanning tree in which edges 4 and 7 have been minimized. Finally, FQ and FH are re-solved with head and flow distributions as shown in Table 3 (FQ(2) and FH(2), respectively). A local optimum is defined at this stage due to the similarity between the last two flow distributions.

Examining the optimal head distribution it is seen that the maximum head constraint becomes active at node 2 which is next to the pump station. On the other hand, the minimum heads are found at nodes 5 and 6 which are at the 'low end' of the flow directions scheme. The diameter of the impeller is determined by substituting the head gained by the pump (69.3 m) into Eq. (39). Its value equals 29.6 cm, and the pump is operated at its most efficient working point. The optimal level of the reservoir is 38.3 m above the ground and the total cost of the network is 309,980 units.

Clearly, a complete solution of a pump-reservoir system should consider more than just one load pattern. Expanding the decomposition technique for several load patterns can be accomplished by the extension of the constraint sets. Our attempts to solve the above network for two load patterns were not successful. We found that extension of the constraint sets raises difficulties. The first is that the feasible region is reduced in size, and therefore it is difficult to find a feasible initial point. Furthermore, the reduced feasible region apparently makes the solution obtained more dependent on the initial point selected. Adding dummy variables such as artificial valves and pumps with a high penalty cost (Alperovits and Shamir[10]) is not of much help. This is due to the fact that the high penalties become the dominant factors in the search procedure, i.e., the minimum solution is concerned more about the penalties than about the actual cost of the network. These issues need further analysis and investigation.

SUMMARY AND CONCLUSIONS

The mathematical model of optimal design of water supply networks is characterized by the following properties: (1) Two independent sets of variables, (2) An objective function which becomes separable after fixing one of the variable sets, (3) A linear set of constraints. The solution proposed in this study exploits the first property by decomposing the model into two submodels, one for each set of variables. The two submodels are solved iteratively and are rapidly convergent to a local optimum. Although not fully proven, the solution is most likely to be achieved after two iterations. The second and the third properties of the mathematical model are exploited via implementation of efficient algorithms: minimum cost flow for the first submodel and linear programming for the second submodel. The use of an explicit equation for a variable speed pump leads to better modeling of the design problem. In addition, the pump is always selected such that it operates at its most efficient working point. The general pump equation is of great potential in adjusting the pump speed for variable operation conditions in an existing network.

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