WATER RESOURCES RESEARCH, VOL. 25, NO. 7, PAGES 1469–1480, JULY 1989

Analysis of the Linear Programming Gradient Method for Optimal Design of Water Supply Networks

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A theoretical analysis of the linear programming (LP) gradient method for optimal design of water distribution networks is presented. The method was first proposed by A. Alperovits and U. Shamir (1977) and has received much attention in the last 10 years. It consists of two stages that are solved in alteration: (1) a LP problem is solved for a given feasible flow distribution and (2) a search is conducted in the space of flow variables, based on the gradient of the objective function (GOF). In this paper a matrix formulation is given for both stages using well-known graph theory matrices. It is proven that the mathematical expression of the GOF is independent of the choice of the sets of loops and paths along which the head constraints are formulated. This is contrary to the claim made by I. C. Goulter et al. (1986). The original GOF expression is shown to have been an approximation of the steepest direction, but still gives good results. Finally, the search procedure is improved by using the projected gradient method.

INTRODUCTION

Optimal design of water supply networks must consider various aspects such as hydraulics, standards of service, reliability, water quality, and consumption patterns. At present, the major optimization techniques handle the following basic problem: given the network layout and a few loading (consumption) patterns, what is the network capacity (pipe diameters) under which the cost is minimal. The reliability and quality of service considerations are incorporated by the requirement that the network be looped. Even for this basic problem the solution is quite complicated due to the nonlinear flow-head loss relationships and the presence of discrete variables, such as commercial pipe diameters. Recently, some attempts have been made to expand the basic problem by considering optimal extension of an existing network, different locations of fires or pipe breaks, and large-scale networks. These extensions are generally based on the previous techniques used to solve the basic problem.

A prominent group of such techniques is one that decomposes the problem into two stages. In the first stage part of the variables are kept constant while the others are solved by linear programming. In the second stage a search technique is employed, which changes the rest of the variables. The stages are solved iteratively until some convergence criterion is met. Decomposition is found attractive because with part of the variables kept constant, the design problem becomes linear and so LP technique may be applied. Works belonging to this procedure are *Alperovits and Shamir* [1977], *Quindry et al.* [1981], *Saphir* [1983], and *Fujiwara et al.* [1987]. Corrections were suggested by *Quindry et al.* [1979] and *Goulter et al.* [1986].

The present work considers the particular method, called linear programming gradient (LPG), originally presented by *Alperovits and Shamir* [1977]. In the first stage a set of flows throughout the network is given and the corresponding optimal set of heads is obtained by LP. In the second stage flows are modified according to the gradient of the objective function (GOF) with respect to the flows. An improvement to the basic LPG method was suggested by *Saphir* [1983] and *Fujiwara et al.* [1987]: following the evaluation of the GOF the flow distribution is changed by a quasi-Newton method instead of a simple gradient search method. *Quindry et al.* [1981] presented an analogous approach in which the first stage is solved by LP for a given set of hydraulic heads. The GOF is determined with respect to these hydraulic heads. In the second stage the set of heads is changed in the direction of the GOF and a new iteration begins.

The most ambiguous part of the LPG methods is the GOF evaluation. The expression for the GOF consists of several terms; each of them is a summation over a group of arcs. In addition, the signs in front of each term depend on the relative direction of the flow with regard to a selected positive sense assumed in each circuit. Since the original publication of the LPG method [*Alperovits and Shamir*, 1977] two comments concerning the gradient expression have been published. The first, by *Quindry et al.* [1979], corrects the original expression by taking into account the interaction between loops and paths included in the head constraints. The second comment by *Goulter et al.* [1986] claims that the solution depends on the choice of paths and loops describing the head constraints. According to this claim, the final discharge in each arc increases with the number of times each arc appears in the set of head constraints, probably due to greater "weight" in the solution.

Following the attempts to use the LPG method for extended design problems it was decided to examine in detail the method and to clarify its theoretical basis. This paper includes the following sections: (1) review of the basic LPG method, (2) derivation of matrix formulation of the problem, (3) refutation of the claim of *Goulter et al.* [1986], (4) reformulation of the constraint set independent of the sets of paths and loops which define the head loss constraints, (5) analysis of the gradient of the objective function (GOF), (6) an illustrative example, and (7) presentation and demonstration of the projected gradient method.

Hereafter, we shall adopt the convention of capital bold letters for matrices and a column, or a row of the matrix.

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Paper number 89WR00428.
0043-1397/89/89WR-00428$05.00

Also, we shall use the vector derivation rule, by which the derivative of a scalar by a vector is the transpose of the derivative vector.

REVIEW OF THE BASIC LPG METHOD

The LPG method [*Alperovits and Shamir*, 1977] is an iterative procedure consisting of two stages. The first (lower) stage optimally solves a looped network with a known flow distribution. The solution is obtained by means of LP, since with a known flow distribution the optimization model becomes linear (except for some minor nonlinearities, which can be linearized). The second (upper) stage improves the flow distribution based on a local GOF. The GOF is calculated using the dual values of the first stage linear model. The process continues iteratively until it converges to a (possibly local) optimum.

For a gravity fed network, designed for a single loading (set of node demands), the first stage of the LPG model is

P1

$$\text{cost} = \min \left\{ \sum_{e \in E} \sum_{d \in D_e} c_{e,d} x_{e,d} \right\} \tag{1}$$

subject to

$$\sum_{e \in l} \sum_{d \in D_e} \pm j_{e,d} x_{e,d} = 0 \qquad \forall l \in L \tag{2}$$

$$\sum_{e \in P} \sum_{d \in D_e} \pm j_{e,d} x_{e,d} \leq \Delta h_p \qquad \forall p \in P \tag{3}$$

$$\sum_{d \in D_e} x_{e,d} = a_e \qquad \forall e \in E \tag{4}$$

$$x_{e,d} \geq 0$$

where

a_e length of arc e;
$c_{e,d}$ cost per unit length of a pipe segment with diameter d in arc e;
$j_{e,d}$ hydraulic gradient on the segment with diameter d in arc e, with the given flow;
$x_{e,d}$ length of a pipe segment with diameter d in arc e;
Δh_p head difference allowed between the ends of path p.

The various sets are $d \in D_e$, set of possible commercial pipe diameters in arc e; $e \in E$, set of all arcs in the network; $l \in L$, set of basic loops (an arbitrary positive direction is selected for each loop); and $p \in P$, set of paths included in the head constraints.

Equation (1) is the objective function, namely, the minimum cost of the pipeline network. In the more general case the cost of pumps, pumping energy, valves, and other equipment is also included. Equation (2) assures that the summation of head losses around each loop is zero. The set of loops L is the set of basic loops such that if (2) holds for L it holds for any other loop. In case of m source nodes the set of loops is increased by $(m - 1)$ paths connecting pairs of sources. Equation (3) is imposed to assure adequate pressure at selected nodes where the paths $p \in P$ connect these nodes with a "reference node" at which the head is known (usually a supply node). The plus or minus sign in (2) and (3) is plus when the flow direction in an arc coincides with the positive sense selected for the path or loop and minus when they oppose. The hydraulic gradient $j_{e,d}$ is calculated using the Hazen-Williams equation:

$$j_{e,d} = \alpha chw_{e,d}^{-1.852} d^{-4.87} q_e^{1.852} \qquad \forall d \in D_e \tag{5}$$

where α is a numerical coefficient whose value depends on the units used, $chw_{e,d}$ is the Hazen-Williams coefficient of the pipe smoothness, and q is the flow. Hereafter we shall refer to (2) and (3) as the loop and path constraints, respectively, while (4) represents the length constraints.

The second stage of the LPG method changes the flow distribution in a way that will reduce the optimal value of the objective function (1). The change of flow must satisfy continuity at all nodes; therefore only a change of circular flows is permitted (or flow along path that connects two sources). Such a change of flows means a uniform flow addition (positive or negative) to all arcs included in a basic loop. According to the original method [*Alperovits and Shamir*, 1977], the amount of flow addition around a specific loop is made proportional to the GOF magnitude with respect to that particular loop flow.

The GOF expression, considering the correction made by *Quindry et al.* [1979, equation] and leaving in the 1.852 coefficient, is

$$\frac{d\,\text{cost}}{dq_{l(i)}} = -\pi_{l(i)} \sum_{e \in l(i)} 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} + \sum_{g \in G} \pi_g \sum_{\substack{e \in G \\ e \in l(i)}} \pm 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} \tag{6}$$

where

$g \in G$ set of paths and loops (excluding loop i) with one or more arcs in common with loop i;
$q_{l(i)}$ circular flow in loop i;
$\Delta h_{e,d}$ head loss along pipe segment in arc e with diameter d ($h_{e,d} = j_{e,d} x_{e,d}$);
$\pi_{l(i)}$ the dual value corresponding to the $l(i)$ loop constraint;
π_g the dual value corresponding to the gth loop/path constraint.

The plus or minus sign in the second term becomes minus when the positive direction selected for any path/loop coincides with the positive direction selected for loop i. Notice that the $d\,\text{cost}/dq_l > 0$ means that a $dq_l < 0$ will reduce the cost function. The same expression for the GOF was recently derived by *Fujiwara et al.* [1987] using the sensitivity theorem.

MATRIX FORMULATION OF THE LPG METHOD

The LPG model (P1) and the GOF expression (equation (6)) can be simplified by means of circuit and path matrices. The terms loop and circuit are used here interchangeably, meaning a sequence of arcs whose starting node is the same as its terminal node. Given a looped pipeline network with (ne) arcs and (nn) nodes, the hydraulic head constraints are represented by (nl) loops and (np) paths; (nd) commercial diameters are assigned to each arc so that the total number of pipe segments is $ns = nd \times ne$. Assuming a feasible flow distribution (one which satisfies continuity at all nodes), the head loss distribution is given by the vector $\mathbf{\Delta h}$, which has (ne) components:

$$\Delta \mathbf{h} = \bar{\mathbf{I}}\mathbf{J}\mathbf{x} \tag{7}$$

where **x** is a vector (size *ns*) of the pipe segment length and **J** is the gradient matrix (size $ns \times ns$) which attaches a hydraulic gradient to each pipe segment (i). The gradient matrix is defined as follows:

$$\begin{aligned} J_{ij} &= \alpha\, chw_i^{-1.852} d_i^{-4.87} q_i^{1.852} \qquad &\text{for } i = j \\ J_{ij} &= 0 &\text{otherwise} \end{aligned} \tag{8}$$

$\bar{\mathbf{I}}$ is a matrix (size $ne \times ns$) which represents the internal arrangement of the pipe segments (j) within the set of arcs (i). The matrix $\bar{\mathbf{I}}$ is defined as follows:

$$\begin{aligned} \bar{I}_{ij} &= 1 \qquad &\text{for } (i-1)nd < j \le (i)nd \\ \bar{I}_{ij} &= 0 &\text{otherwise} \end{aligned} \tag{9}$$

The matrix form of the loop constraints is given by

$$\mathbf{L}\Delta\mathbf{h} = \mathbf{0} \tag{10}$$

where **L** is the circuit matrix (size $nl \times ne$), whose rows and columns correspond to the basic loops and arcs, respectively. Define a positive direction of circulation for each basic loop, the terms of **L** are

$$\begin{aligned} L_{ij} &= +1 \qquad &\text{arc } j \text{ is in loop } i\text{, same sense} \\ L_{ij} &= -1 &\text{arc } j \text{ is in loop } i\text{, opposite sense} \\ L_{ij} &= 0 &\text{arc } j \text{ is not in loop } i \end{aligned} \tag{11}$$

The matrix form of path constraints is given by

$$\mathbf{P}\Delta\mathbf{h} \le \Delta\mathbf{h}_p \tag{12}$$

where $\Delta\mathbf{h}_p$ is the vector (size *np*) of the maximum admissible headlosses along (*np*) paths connecting reference nodes with the other nodes. **P** is the path matrix (size $np \times ne$) whose rows and columns correspond to the paths and arcs, respectively. Define a positive direction for each path, the terms of the path matrix are

$$\begin{aligned} P_{ij} &= +1 \qquad &\text{arc } j \text{ is in the path } i\text{, same sense} \\ P_{ij} &= -1 &\text{arc } j \text{ is in the path } i\text{, opposite sense} \\ P_{ij} &= 0 &\text{arc } j \text{ is not in the path } i \end{aligned} \tag{13}$$

Based on the above equations it is now possible to reformulate the basic LPG model in its matrix form:

P2

$$\text{cost}\ (\mathbf{q}) = \min\ \{\mathbf{c}^T\mathbf{x}\} \tag{14}$$

subject to

$$\mathbf{L}\bar{\mathbf{I}}\mathbf{J}(\mathbf{q})\mathbf{x} = \mathbf{o} \tag{15}$$

$$\mathbf{P}\bar{\mathbf{I}}\mathbf{J}(\mathbf{q})\mathbf{x} \le \Delta\mathbf{h}_p \tag{16}$$

$$\bar{\mathbf{I}}\mathbf{x} = \mathbf{a} \tag{17}$$

$$\mathbf{x} \ge \mathbf{o} \tag{18}$$

where **c** is the cost vector (size *ns*), which is the cost per unit of length of each pipe segment, and **a** is the vector of the arc lengths (size *ne*). Next we shall develop the expression for the GOF, starting from the corrected version as suggested by *Quindry et al.* [1979, equation 6]. Separating the set G into the set of loops L and the set of paths P, (6) becomes

$$\begin{aligned} d\,\text{cost}/dq_{l(i)} = &- \pi_{l(i)} \sum_{e \in l(i)} 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} \\ &+ \sum_{\substack{l \in L \\ l \ne l(i)}} \pi_l \sum_{\substack{e \in l(i) \\ e \in l}} \pm 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} \\ &+ \sum_{p \in P} \pi_p \sum_{\substack{e \in l(i) \\ e \in p}} \pm 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} \end{aligned} \tag{19}$$

To make our notation consistent with that commonly used in graph theory, a notation which is also more intuitively appealing, we adopt the following convention: the sign in (19) is made plus when the direction of the arc coincides with the positive direction of the path/loop. Combining the first two right-hand side terms and using the suggested notation, (19) becomes

$$\begin{aligned} d\,\text{cost}/dq_{l(i)} = &- \sum_{l \in L} \pi_l \sum_{\substack{e \in l \\ e \in l(i)}} \pm 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} \\ &- \sum_{p \in P} \pi_p \sum_{\substack{e \in p \\ e \in l(i)}} \pm 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} \end{aligned} \tag{20}$$

Based on the Hazen-Williams equation (equation (5)), it follows that

$$\begin{aligned} 1.852 q_e^{-1} \sum_{d \in D_e} \Delta h_{e,d} &= \sum_{d \in D_e} \frac{1.852 \Delta h_{e,d}}{q_e} \\ &= \sum_{d \in D_e} \frac{\partial \Delta h_{e,d}}{\partial q_e} = \frac{\partial \Delta h_e}{\partial q_e} \end{aligned} \tag{21}$$

and so (20) becomes

$$\frac{d\,\text{cost}}{dq_{l(i)}} = \sum_{l \in L} \pi_l \sum_{\substack{e \in l \\ e \in l(i)}} \pm \frac{\partial \Delta h_e}{\partial q_e} - \sum_{p \in P} \pi_p \sum_{\substack{e \in p \\ e \in l(i)}} \pm \frac{\partial \Delta h_e}{\partial q_e} \tag{22}$$

Similar to the admittance matrix in electrical network a new matrix **S** (size $ne \times ne$) is defined, called the "resistance matrix" of a pipeline network:

$$\mathbf{S} = \partial\Delta\mathbf{h}/\partial\mathbf{q} \quad \text{or} \quad \begin{aligned} S_{ij} &= \partial\Delta h_i/\partial q_j \qquad &\text{for } i = j \\ S_{ij} &= 0 &\text{otherwise} \end{aligned} \tag{23}$$

S is a diagonal matrix whose elements are always nonnegative and are independent on the assumed flow distribution. Since **L** and **P** contain the proper signs for the various terms, (22) can be rewritten in matrix form

$$d\,\text{cost}/dq_{l(i)} = -\boldsymbol{\pi}_l^T \mathbf{L}\mathbf{S}L_i^T - \boldsymbol{\pi}_p^T \mathbf{P}\mathbf{S}L_i^T \tag{24}$$

where L_i^T is the transpose of the ith row in the circuit matrix and $\boldsymbol{\pi}_l^T$ and $\boldsymbol{\pi}_p^T$ are the transposes of the dual vectors corresponding to the loop and path constraints, respectively. Finally, the GOF expression is obtained by combining the two right-hand side terms and extending (24) for all circular flows:

$$\text{GOF} = d\,\text{cost}/d\mathbf{q}_l = -\boldsymbol{\pi}^T(\mathbf{L}/\mathbf{P})\mathbf{S}\mathbf{L}^T \tag{25}$$

where $\boldsymbol{\pi}^T = (\boldsymbol{\pi}_l^T | \boldsymbol{\pi}_p^T)$. The resulting expression can be easily computed, since the matrices **L** and **P** consist of only (± 1) and (0) elements, while matrix **S** is diagonal.

The GOF is used to make a change in the flow distribution. *Alperovits and Shamir* [1977] and *Quindry et al.* [1981] make a simple gradient move. *Saphir* [1983] and *Fujiwara et al.* [1987] use a quasi-Newton method, with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update procedure, which improves considerably the algorithm's performance.

THE CHOICE OF LOOP AND PATH CONSTRAINTS FOR THE LPG METHOD

Next we examine the influence of different choices of the loop and path constraints required for the LPG method. It was claimed by *Goulter et al.* [1986] that different choices may result in different optimal solutions, due to a change in the GOF values. In order to prove the contrary, namely, the independence of the GOF on loop or path set, it is first necessary to express the GOF independently of a particular set of circular flows. Therefore the GOF (equation (25)) expressed for circular flows, $d\ \text{cost}/d\mathbf{q}_l$, is converted into an equivalent expression, $d\ \text{cost}/d\mathbf{q}$, which is the GOF with respect to arc flows in the network. The relationship between the set of flows $\mathbf{q}$ and the set of circular flow $\mathbf{q}_l$ is given by

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{L}^T\mathbf{q}_l \tag{26}$$

where q_0 is some arbitrary feasible set of flows (i.e., satisfying continuity at all nodes). From (26) it follows that

$$d\mathbf{q}/d\mathbf{q}_l = \mathbf{L}^T \tag{27}$$

Using the chain rule for $d\ \text{cost}/d\mathbf{q}_l$ and substituting (27) results in

$$\text{GOF} = \frac{d\ \text{cost}}{d\mathbf{q}_l} = \frac{d\ \text{cost}}{d\mathbf{q}}\frac{d\mathbf{q}}{d\mathbf{q}_l} = \frac{d\ \text{cost}}{d\mathbf{q}}\mathbf{L}^T \tag{28}$$

Comparing this result with (25) it is seen that

$$d\ \text{cost}/d\mathbf{q} = -\boldsymbol{\pi}^T(\mathbf{L/P})\mathbf{S} \tag{29}$$

We propose to prove that (29) gives the same result for any chosen set of loop and path constraints.

Consider a looped network with (nn) nodes and (ne) directed arcs. Based on graph theory considerations it can be shown that if the loop constraints hold for a set of basic loops it also holds for any other loop [*Deo*, 1974, p. 212]. A basic loop (circuit) is defined by the addition of a single arc (chord) to a spanning tree of the network. The set of basic loops is therefore the set of all basic loops with respect to a particular spanning tree. There are $(ne - nn + 1)$ independent basic loops in a network. Regarding a basic loop as a row vector of the loop matrix $\mathbf{L}$, it follows that given a connected looped network, any loop can be described as a linear combination of the rows of the loop matrix $\mathbf{L}$. As for the path constraints, one has to select a set of paths connecting the reference node, at which the head is known (usually the source node), with the other nodes at which the head is to be kept above a specific minimum. For a connected looped network there are one or more paths between each pair of nodes and it can be easily shown that a linear combination of given path and a proper set of loops can define any other path between the same pair of nodes.

Linear combinations of matrix rows can be performed by elementary operations, defined as adding (substracting) k times row i to (from) row j. Given the matrix of the head constraints $(\mathbf{L/P})_0$, the new matrix $(\mathbf{L/P})_n$, resulting from such an elementary operation, is

$$(\mathbf{L/P})_n = \mathbf{E}(\mathbf{L/P})_0 \tag{30}$$

$$\mathbf{E} = \mathbf{I} + k\mathbf{e}_j\mathbf{e}_i^T \tag{31}$$

where $\mathbf{E}$ is an elementary matrix, size $(ne \times ne)$, $\mathbf{I}$ is a unit matrix, size $(ne \times ne)$, and $\mathbf{e}_i$ is a unit vector with (± 1) in the ith entry and (0) elsewhere, size (ne). Note that if the GOF is proved to be independent for one elementary operation, it is obviously independent for a series of elementary operations.

The elementary matrix used to convert the circuit and path matrix may also be used to express the change in the dual vector π, due to a change in the circuit and path constraints. The initial vector is given by

$$\pi_o^T = \mathbf{c}_b^T\mathbf{A}_b^{-1} \tag{32}$$

where $\mathbf{A}_b$ is the optimal basis of the constraint matrix and $\mathbf{c}_b$ is the corresponding cost vector. Since the basis does not change by elementary operations, the dual vector following an elementary operation is

$$\boldsymbol{\pi}_n^T = \mathbf{c}_b^T(\mathbf{E}\mathbf{A}_b)^{-1} = \mathbf{c}_b^T\mathbf{A}_b^{-1}\mathbf{E}^{-1} = \boldsymbol{\pi}_o^T\mathbf{E}^{-1} \tag{33}$$

where the indices (o) and (n) come for the new and old dual vector, respectively.

To complete the proof it is shown that the GOF value remains unchanged after an elementary operation. Following an elementary operation the new GOF is

$$(d\ \text{cost}/d\mathbf{q})_n = -\boldsymbol{\pi}_n^T(\mathbf{L/P})_n\mathbf{S} \tag{34}$$

Substituting (30) and (33) into (34) results in

$$\left(\frac{d\ \text{cost}}{d\mathbf{q}}\right)_n = -\boldsymbol{\pi}_o^T\mathbf{E}^{-1}\mathbf{E}\left(\frac{\mathbf{L}}{\mathbf{P}}\right)_o\mathbf{S} = -\boldsymbol{\pi}_o^T\left(\frac{\mathbf{L}}{\mathbf{P}}\right)_o\mathbf{S} = \left(\frac{d\ \text{cost}}{d\mathbf{q}}\right)_o \tag{35}$$

that is, the GOF value before and after a change in the loop/path constraints is unchanged. The theoretical result has also been demonstrated in our computer runs. We surmise that *Goulter et al.* [1986] may have had an error in their algorithm or in the computations.

REFORMULATION OF THE HEAD CONSTRAINTS

An alternative formulation of the head constraints which alleviates the need to select paths and loops, is based on the following consideration: the head loss along any arc is equal to the hydraulic head difference between its two end nodes. Such a constraint is mathematically expressed by

$$\bar{\mathbf{R}}^T\mathbf{h} - \boldsymbol{\Delta}\mathbf{h} = \mathbf{o} \tag{36}$$

where $\mathbf{h}$ (size nn) is the vector of the hydraulic heads and $\boldsymbol{\Delta}\mathbf{h}$ (size ne) is the vector of the head losses. The matrix $\bar{\mathbf{R}}$ (size $nn \times ne$) is a well-known matrix in graph theory, called the incidence matrix. The elements of the incidence matrix are defined as follows:

$$\begin{aligned} \bar{R}_{ij} &= +1 && \text{if arc } j \text{ is directed away from node } i \\ \bar{R}_{ij} &= -1 && \text{if arc } j \text{ is directed toward node } i \\ \bar{R}_{ij} &= 0 && \text{otherwise} \end{aligned} \tag{37}$$

Each column of $\bar{\mathbf{R}}$ corresponds to an arc and has exactly two nonzero entries, one being $(+1)$ and the other (-1). Each row in $\bar{\mathbf{R}}$ corresponds to a node, and the nonzero entries indicate the arcs incident to that node. Since the rank of $\bar{\mathbf{R}}$ is $(nn - 1)$, [*Deo*, 1974, p. 139] it is possible to eliminate one row without loss of information. Eliminating the row of the supply node R_s, the reduced incidence matrix $\mathbf{R}$ is implicitly given by

$$\bar{\mathbf{R}} = (R_s/\mathbf{R}) \tag{38}$$

Define a new set of variables $\mathbf{h}^+$ to be the surplus head at each demand node with respect to its minimum allowed head $h_{\min}$; (36) becomes

$$(R_s^T|\mathbf{R}^T)(h_s/(\mathbf{h}^+ + \mathbf{h}_{\min})) - \mathbf{\Delta h} = \mathbf{o} \tag{39}$$

where h_s denotes the constant head at the supply node. Rearranging the terms in (39) it follows that

$$\mathbf{R}^T\mathbf{h}^+ - \mathbf{\Delta h} = -R_s^T h_s - \mathbf{R}^T\mathbf{h}_{\min} \tag{40}$$

Since the summation of all rows of $\bar{\mathbf{R}}$ always gives $\mathbf{o}$, it can be shown that

$$\mathbf{R}^T\mathbf{h}_s = -R_s^T h_s \qquad \text{where} \quad \mathbf{h}_s = [h_s, h_s, \cdots, h_s]^T \tag{41}$$

and so (40) becomes

$$\mathbf{R}^T\mathbf{h}^+ - \mathbf{\Delta h} = \mathbf{R}^T(\mathbf{h}_s - \mathbf{h}_{\min}) = \mathbf{R}^T\mathbf{\Delta h}_p \tag{42}$$

where $\mathbf{\Delta h}_p$ is the hydraulic head difference between the supply node and the demand nodes.

Using (42) for the head constraints the LPG first stage can be reformulated as follows:

P3

$$\text{cost} = \min \{\mathbf{c}^T\mathbf{x}\} \tag{43}$$

Subject to

$$\mathbf{R}^T\mathbf{h}^+ - \bar{\mathbf{I}}\mathbf{J}\mathbf{x} = \mathbf{R}^T\mathbf{\Delta h}_p \tag{44}$$

$$\bar{\mathbf{I}}\mathbf{x} = \mathbf{a} \tag{45}$$

$$\mathbf{h}^+, \mathbf{x} \geqslant \mathbf{o} \tag{46}$$

P3 and P2 are mathematically equivalent and therefore result in the same optimal solution. A proof of the equivalence is given in Appendix A where P2 is converted into P3 by a series of elementary operations. Comparing the two formulations it is seen that P3 combines the path and the loop constraints into a single type of constraint, while the total number of constraints is not changed. P3 has the advantage of defining a uniform set of head constraints independent of the choice of the paths and loops in P2. P3 also simplifies the programming procedure, since it requires the incidence matrix only, as compared to the path and the loop matrices of P2.

Analysis of the GOF Expression

The second stage of the LPG method changes the circular flows q_l in a way that reduces the value of objective function. The change is made according to the gradient of objective function, named GOF, with respect to the circular flows. Having the GOF value, the circular flows are modified according to either (1) The steepest descent direction, in which the search direction is parallel to that of the GOF [*Alperovits and Shamir*, 1977], or (2) A quasi-Newton search direction, in which the GOF is used to update a second-order approximation of the Hessian matrix by the BFGS method [*Saphir*, 1983; *Fujiwara et al.*, 1987].

A general model unifing the two stages of the LPG method may be formulated as follows:

P4

$$\min \{\text{cost}(\mathbf{q}_l)\} \tag{47}$$

$$\text{cost}(\mathbf{q}_l) = \min \{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}(\mathbf{q}_l)\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{o}\} \tag{48}$$

where $\text{cost}(\mathbf{q}_l)$ is the objective function and $\mathbf{A}(\mathbf{q}_l)$ is the constraint matrix consisting of the loop, path, and length constraints. The original inequality constraints are converted into equalities by adding slack variables. The elements of $\mathbf{A}$ depend on the circular flow vector $\mathbf{q}_l$, due to the hydraulic gradient which appears in the path and loop constraints. Denote the optimal value of $\mathbf{x}$, for a particular value of $\mathbf{q}_l$, by $\mathbf{x}^*$, P4 becomes

$$\min \{\text{cost}(\mathbf{q}_l) = \mathbf{c}_b^T\mathbf{x}_b^*\} \tag{49}$$

where the subscript b denotes the basis of $\mathbf{x}^*$. Since the optimal basis is uniquely defined by the optimal selection of the nonbasic variables, $\mathbf{x}_n^*$, the objective function should be viewed as a function of $\mathbf{q}_l$ and $\mathbf{x}_n^*$:

$$\text{cost}(\mathbf{q}_l) = \mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*) \tag{50}$$

Applying the chain rule for the gradient of the objective function, we have

$$\text{GOF} = \frac{d\ \text{cost}(\mathbf{q}_l)}{d\mathbf{q}_l} = \frac{d\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)}{d\mathbf{q}_l} = \frac{\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{q}_l} + \frac{\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{x}_n^*}\frac{d\mathbf{x}_n^*}{d\mathbf{q}_l} \tag{51}$$

The terms on the right-hand side of the (27) are analyzed, term by term as follows:

$$\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial\mathbf{q}_l$$

This term is responsible for the change in the optimal value of the objective function, caused by a small change in the circular flow, where the set of nonbasic variables is kept unchanged. As a result of such a change, the basic variables take on new (optimal) values. The basic variables are the length of the pipe segments which are included in the optimal solution, while the nonbasic variables are the pipe segments rejected by the optimal solution (i.e., lengths equal to zero). A full evaluation of the above derivative is given in Appendix B. An intermediate result is

$$\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial\mathbf{q}_l = -\boldsymbol{\pi}^T\ \partial\mathbf{b}(\mathbf{q}_l, \mathbf{x}^*)/\partial\mathbf{q}_l \tag{52}$$

and the final result corresponding to the original LPG formulation (model P2) is

$$\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial\mathbf{q}_l = -\boldsymbol{\pi}^T(\mathbf{L}/\mathbf{P})\mathbf{S}\mathbf{L}^T \tag{53}$$

which is identical to the original GOF expression [*Alperovits and Shamir*, 1977], given by (25). For the alternative formulation (model P3) the final result is

$$\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial\mathbf{q}_l = \boldsymbol{\pi}^T\mathbf{S}\mathbf{L}^T \tag{54}$$

which is simpler compared with the previous expression.

$$\{\partial \mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial \mathbf{x}_n^*\}\{d\mathbf{x}_n^*/d\mathbf{q}_l\}$$

This term is responsible for the change in the optimal value of the objective function caused by a change of the nonbasic variables. The first derivative is known as the reduced gradient of the LPG's first stage [*Reklaitis et al.*, 1983] and is given by

$$\partial \mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial \mathbf{x}_n^* = \mathbf{c}_n^T - \mathbf{c}_b^T \mathbf{A}_b^{-1} \mathbf{A}_n = \mathbf{c}_n^T - \boldsymbol{\pi}^T \mathbf{A}_n \geq \mathbf{o} \quad (55)$$

where $\mathbf{A}_b$ and $\mathbf{A}_n$ are the submatrices of $\mathbf{A}$ corresponding to the basic and the nonbasic variables, respectively. The inequality sign expresses the LP necessary condition under which $\mathbf{x}$ becomes optimal.

The second derivative $d\mathbf{x}_n^*/d\mathbf{q}_l$ cannot be evaluated using only results of the first stage of the LPG. Still, we can show that $d\mathbf{x}_n^*/d\mathbf{q}_l$ is not zero under all circumstances. Consider an optimal solution of P2 for a given flow distribution. Each arc consists of at most two pipe segments, regardless of the number of candidate pipe diameters [*Orth*, 1986, p. 155]. In order to have $d\mathbf{x}_n^*/d\mathbf{q}_l = \mathbf{o}$, it is necessary that the same variables remain in the basis and the nonbasic variables $\mathbf{x}_n^*$ remain equal to zero, under the small change of flow $d\mathbf{q}_l$. If we want to guarantee that no new pipe segments will enter the basis due to the change $d\mathbf{q}_l$, then all pipes must have two segments in the optimal solution (and not one, as many happen) so that their relative lengths are modified as the change $d\mathbf{q}_l$ is imposed. This means that $(2 \times ne)$ pipe segments appear in the basis, whose size is also $(2 \times ne)$, equal to the number of constraints. It follows that all remaining variables which are the head surpluses $\mathbf{h}^+$ do not appear in the basis, i.e., $\mathbf{h}^+ = 0$. Such a case where all heads throughout the network are at their minimum is not practically possible. We therefore conclude that $d\mathbf{x}_n^*/d\mathbf{q}_l$ is not always zero.

Summerizing the above analysis, the GOF is given by

$$\text{GOF} = d\ \text{cost}/d\mathbf{q}_l = -\boldsymbol{\pi}^T\ \partial\mathbf{b}(\mathbf{q}_l, \mathbf{x}^*)/\partial\mathbf{q}_l + (\mathbf{c}_n^T - \boldsymbol{\pi}^T\mathbf{A}_n)\ d\mathbf{x}_n^*/d\mathbf{q}_l \quad (56)$$

where the first term equals the original GOF, suggested by *Alperovits and Shamir* [1977], but the second term may be nonzero. As a result, the original GOF is somewhat different from the exact GOF and should be considered as an approximated gradient. The original GOF gives exact results whenever the change in flow does not cause the introduction of new variables into the basis. Using a steepest descent direction method [*Alperovits and Shamir*, 1977] the GOF is still acceptable, since a new GOF is computed at each iteration, independent of the previous GOF (an illustration is given in the next section). The quasi-Newton method used by *Saphir* [1983] and *Fujiwara et al.* [1987] was shown to be more efficient than the simple steepest descent method, when the approximate GOF (equation (25)) is used. This relative advantage must be reexamined if a more accurate GOF (equation (56)) is to be used.

The resulting GOF holds also for the dual method, proposed by *Quindry et al.* [1981]. Using the same procedure, it can be showed that its GOF also does not account for the optimal change in the nonbasic variables and thus should also be regarded as an approximated expression.

An Illustrative Example

Following the analysis of the GOF expression, it is shown that the GOF consists of two components and that only one of them is computable. The second component, which represents the contribution of the change in the nonbasic variables, can not be evaluated within the framework of the LPG method. It is the purpose of this section to demonstrate that the approximation based on the first component only gives good results. The illustrative example of the LPG performance is based on a simple network, originally presented by *Alperovits and Shamir* [1977].

Consider the network in Figure 1 which is fed by gravity from a constant head reservoir. The demands are given in cubic meters per hour and the minimum head at each node is 30 m above the ground level. There are eight arcs, 1000 m long each, all with Hazen-Williams coefficient equal to 130. Pipe costs are given in arbitrary units in Table 1. The domain of feasible flow distributions is fully described by two circular flows, q_{l1} and q_{l2}, and by the initial flow distribution, shown in Figure 1. Subject to the assumed flow directions, q_{l1} and q_{l2} are limited to changes within a confined domain of triangular shape, described in Figure 2. The edges of the triangle correspond to three arcs in which the flows vanish (4, 7, 8). The corners of the triangle correspond to the spanning trees, each obtained by elimination of two arcs. These spanning trees are drawn at the vertices of the triangle in Figure 2. The response surface, which gives the value of the optimal solution of P1 for each flow distribution, is shown in Figure 2. Here 169 flow distributions on a 10 m^3/hour grid in this domain were used to generate this map, which is shown by lines of constant function value. The surface of cost($\mathbf{q}_l$) looks concave with a global minimum in the vicinity of the lower-left corner. This corner corresponds to the spanning tree obtained after eliminating arcs 4 and 8.

The LPG method was applied using the approximate GOF given by (25). For each iteration the first stage defines a point on the surface of the objective function while the second stage defines a search direction along which the objective function is improved. The step size in the second stage is determined as follows: the circular flow for which the gradient component is maximum is changed by 20 m^3/hour. The other circular flow is changed by 20 m^3/hour times the ratio of its gradient component to the maximum gradient component.

Two starting points are considered in this example. The first is the original case given in the works by *Alperovits and Shamir* [1977], *Quindry et al.* [1979], and *Fujiwara et al.* [1987]. Its starting point, which corresponds to the circular flows of $q_{l1} = 150$ m^3/hour and $q_{l2} = 120$ m^3/hour, appears as point A in the Figure 2. The second case starts from an arbitrary point and corresponds to $q_{l1} = 220$ m^3/hour and $q_{l2} = 160$ m^3/hour, denoted by point D. Since no limitations are set on the minimum and maximum hydraulic gradients, the cost found at point A (475 10^3) is less than the cost found by Alperovits and Shamir (494 10^3) or Fujiwara et al. (487 10^3). The demand for minimum hydraulic gradient is alternatively represented by a minimum discharge of 10 m^3/hour in each arc (shown by dashed lines in Figure 2).

Following the two runs of the LPG performance, from points A and D, respectively, it is seen that the approximate GOF keeps, in general, the steepest direction although some "zig-zag"'s appear. The general direction is kept because in each iteration a new GOF is obtained independent of the previous ones. The two cases end at points B and E, respectively. The fact that the two cases do not end at a true local minimum reveals a major drawback of the original LPG

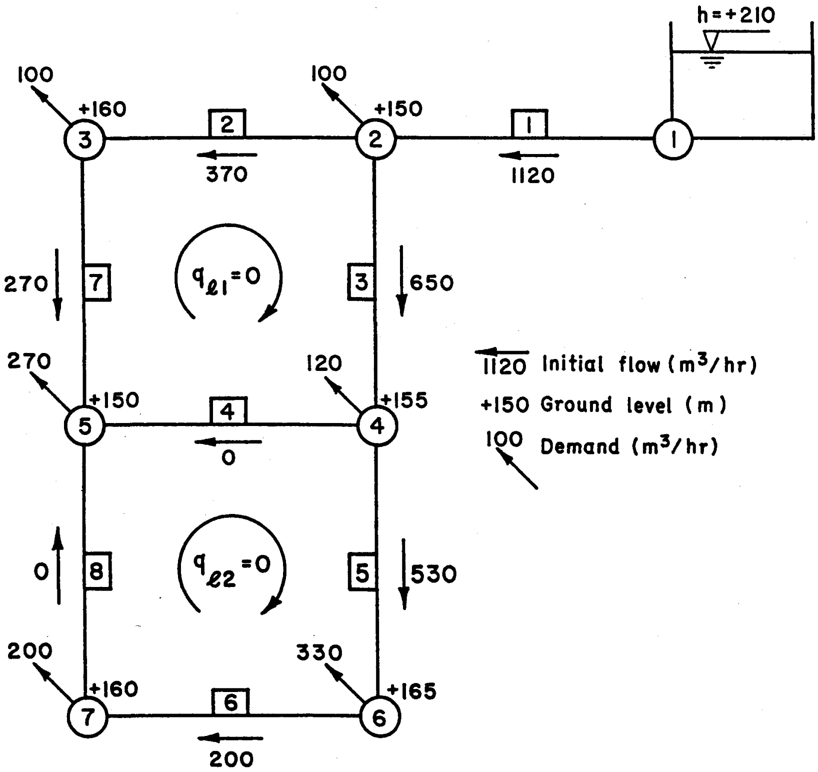

Fig. 1. Simple water supply network (after *Alperovits and Shamir* [1977]).

method. That is, the GOF is an unconstrained gradient method, while actually a constrained gradient method is required. In the next section a correction is suggested based on the gradient projection method.

THE PROJECTED GOF METHOD

The second stage of the LPG method changes the circular flows so that continuity at all nodes is always satisfied. An additional constraint which concerns the flow distribution is that the flow direction in each arc is assumed and is not allowed to change (equivalent to the nonnegativity constraints). Using an unconstrained gradient, this additional constraint is not enforced and so the final result is not necessarily optimal. For example, the line searches in Figure 2 terminate at the edge of the region where arc 8 attains zero flow. No further iterations are executed, since the final GOF points towards negative flow in arc 8.

TABLE 1. Basic Cost Data for Pipes

Diameter, inches	Unit Cost
1	2.0
2	5.0
3	8.0
4	11.0
6	16.0
8	23.0
10	32.0
12	50.0
14	60.0
16	90.0
18	130.0
20	170.0
22	300.0
24	550.0

After *Alperovits and Shamir* [1977]. One inch equals 2.54 cm.

There are two techniques by which the gradient can incorporate constraints. The first is the reduced gradient, used in the GRG method. The second is the projected gradient, where the gradient is projected onto the constraint surface. The two techniques require a similar numerical effort and have the same theoretical background. We have elected to adopt the projected gradient over the reduced gradient due to its geometrical interpretation, although both have similar performances.

The complete set of constraints under which the flows are restricted may be described as follows:

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{L}^T\mathbf{q}_l \tag{57}$$

$$\mathbf{q} \geq \mathbf{q}_{\min} \tag{58}$$

where $\mathbf{q}_0$ is any fixed feasible flow distribution and $\mathbf{q}_{\min}$ is the minimum allowed flow in each arc. A minimum flow constraint stands for the quality of service and reliability demands of water supply network. Substituting (58) into (57) the constraint surface of $\mathbf{q}_l$ is obtained:

$$\mathbf{L}^T\mathbf{q}_l \geq \mathbf{q}_{\min} - \mathbf{q}_0 \tag{59}$$

Corresponding to the triangular region in Figure 2.

Applying the projected gradient method, it is necessary to evaluate the gradient and the maximum step size along it. The projected gradient is evaluated for the following two cases:

1. The present point $\mathbf{q}_l$ is an interior point of the feasible region, so that the flow constraints are all nonactive. In this case the projected gradient is equal to the unconstrained gradient, i.e., the GOF.

2. The present point is on the edge of the feasible region

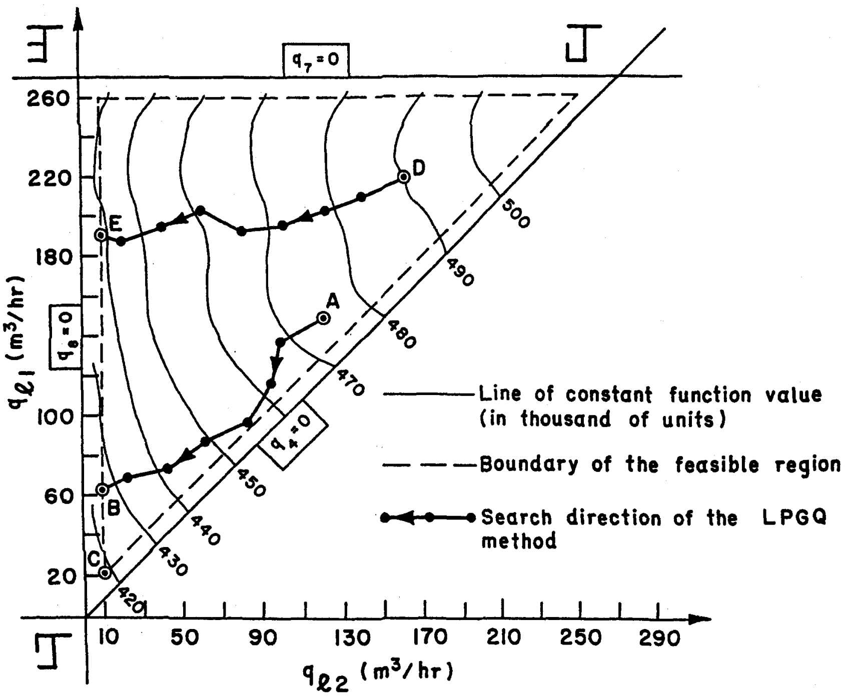

Fig. 2. Objective function surface of the simple network.

so that one or more flow constraints become active. The construction is carried out by decomposing the GOF into two orthogonal components: one parallel to the constraint surface and one perpendicular to it. The parallel component will be the desired vector projection. It can be shown [*Reklaitis et al.*, 1983, p. 407] that the projected gradient is given by

$$(\mathbf{I} - \bar{\mathbf{L}}(\bar{\mathbf{L}}^T\bar{\mathbf{L}})^{-1}\bar{\mathbf{L}}^T)(-\mathrm{GOF}) \tag{60}$$

where $\bar{\mathbf{L}}$ is a submatrix of the loop matrix $\mathbf{L}$, whose columns correspond to the set of active constraints. The maximum step size may be derived by (59) so that the next point will be within the feasible region. A detailed description of the GOF projection algorithm is given in Appendix C, followed by a solution of the previous simple network.

Generally, it is known that the optimal solution tends towards a tree configuration; that is, $(ne - nn + 1)$ of the arcs of zero or minimum flow discharge. From a geometrical point of view these arcs correspond to the edges of the feasible region, while the final solution is a vertex where $(ne - nn + 1)$ edges intersect. Starting from an interior point of the feasible region and proceeding according to the LPG algorithm, one of the constraints becomes active. Applying the projected gradient, the new search direction lies along the edge of the active constraint until another constraint becomes active, etc. It follows that after encountering the first active constraint, the line search is conducted along the edges of the feasible region. The search is terminated when a vertex of $(ne - nn + 1)$ edges is reached. The necessary conditions under which the last vertex is optimal are given and tested by the gradient projection algorithm. The major computational burden of this algorithm consists of the updates of $(\bar{\mathbf{L}}^T\bar{\mathbf{L}})^{-1}$ each time the active constraint set is modified. However, since constraints are added one at a time it is possible to develop formulae for updating $(\bar{\mathbf{L}}^T\bar{\mathbf{L}})^{-1}$ which make use of the prior values of that matrix [*Rosen*, 1960].

Solving the simple network by the projected GOF algorithm (Appendix C), it is shown that the optimal flow distribution is found at a point C (Figure 2), where q_{l1} = 20 m³/hour and q_{l2} = 10 m³/hour. Furthermore, point C satisfies the Kuhn-Tuker conditions for optimality (step 7, Appendix C) and is therefore a local optimum. Note that unless a minimum flow constraint is imposed, arcs 4 and 8 will attain zero flows discharge and so the optimal solution will approach a tree configuration. Comparison of the optimal solutions, which have been obtained so far for this network, is presented in Table 2. It gives the optimal cost and its corresponding optimal flow distribution, given by q_{l1} and q_{l2} with respect to the same initial flow distribution (given in Figure 1). While the previous works restricted the hydraulic gradient to be between 0.5 to 50 promiles, the present work restricts, instead, the minimum flow discharge to 10 m³/hour. Still, the hydraulic gradients of the present solution are all less than 50 promiles so that our solution is comparable to the optimal solutions of others.

Allocating the optimal flow distributions on the constraint surface at Figure 2, it is clear that earlier solutions are away from the local optimum (point C) and thus are not optimal. Comparing between the present solution and the solution obtained by *Fujiwara et al.* [1987], it seems that two different flow distributions give similar costs. However, our solution results in 10 m³/hour in arc 8 while *Fujiwara et al.* [1987] find

TABLE 2. Optimal Solutions of the Simple Water Supply Network

Reference	Cost, units	q_{l1}, m³/hour	q_{l2}, m³/hour
Alperovits and Shamir [1977]	497,525	193.0	71.6
Quindry et al. [1979]	441,552	58.0	5.0
Goulter et al. [1986]	435,015	58.0	0.7
Fujiwara et al. [1987]	415,271	35.68	1.01
Present solution	417,500	20.0	10.0

1.01 m^3/hour in arc 8, which is actually zero and thus impractical from reliability point of view. Relaxing the reliability constraints (i.e., $q \geq 0$ instead of $q \geq 10$ m^3/hour), the minimal cost solution is 400,155 units corresponding to a spanning tree in the vicinity of point C ($q_{l1} = q_{l2} = 0$).

CONCLUSIONS

Each stage of the LPG method was analyzed and reformulated. The first stage was introduced in matrix form and shown to be independent of the choice of the paths and loops, contrary to the claim by *Goulter et al.* [1986]. In the second stage of the LPG method it is shown that the original gradient of the objective function (GOF) did not account for the change of the nonbasic variables. Hence that expression should be regarded as an approximate gradient which, however, has been found to give good results. Due to the iterative process of the LPG the GOF is evaluated each time independently of the previous ones, and so is self-correcting. The flow bound constraints which result from the prefixed flow direction in each arc are taken into account in the second stage by converting the original unconstrained gradient search into a constrained one. This is done by projecting the GOF onto the constraint surface, which requires some additional numeric steps.

APPENDIX A: THE RELATIONSHIP BETWEEN VARIOUS FORMULATIONS OF THE HEAD CONSTRAINTS

The original version of the LPG method describes the head constraints by sets of paths and basic loops. The alternative version describes the head constraint by the incidence matrix, $\bar{\mathbf{R}}$ (equation (37)). We shall prove that both versions are identical and discuss first the relationships between the path matrix and the incidence matrix.

Give a water supply network with of one source node and $(nn - 1)$ demand nodes. The nodes are connected by (ne) directed arcs such that no self-loop or parallel arcs exists. Consider a spanning tree T rooted at the source node, which defines a unique path between the source and each of the demands nodes. This set of $(nn - 1)$ paths is represented by the path matrix $\mathbf{P}$ (size $(nn - 1) \times ne$), defined by (13). Reallocating the arcs of T (called branches) to occupy the first $(nn - 1)$ columns of $\mathbf{P}$, we have

$$\mathbf{P} = [\mathbf{P}_t|\mathbf{o}] \tag{A1}$$

where $\mathbf{P}_t$ (size $(nn - 1) \times (nn - 1)$) is a submatrix of $\mathbf{P}$. Following the same columns arrangement, the reduced incidence matrix $\mathbf{R}$ is given by

$$\bar{\mathbf{R}} = [R_s/\mathbf{R}] \qquad \mathbf{R} = [\mathbf{R}_t|\mathbf{R}_c] \tag{A2}$$

where $\mathbf{R}_t$ is a submatrix of $\mathbf{R}$ (size $(nn - 1) \times (nn - 1)$) consisting of columns which belong to the branches, while $\mathbf{R}_c$ (size $(nn - 1) \times (ne - nn + 1)$) consists of columns which belong to the rest of the arcs (called chords).

Consider a demand node i and its path to the root p_i given by the ith row of $\mathbf{P}$. Then

$$\bar{\mathbf{R}}P_i^T = \mathbf{e}^s - \mathbf{e}^i \tag{A3}$$

where $\mathbf{e}^s$ and $\mathbf{e}^i$ are the unit vectors (size ne):

$$\mathbf{e}^s = [1, 0, 0, \cdots, 0]^T \qquad \mathbf{e}^i = [0, \cdots, 0, 1, 0, \cdots, 0]^T \tag{A4}$$

The left-hand side of (A3) is a summation of columns of $\bar{\mathbf{R}}$, corresponding to the arcs which appear in the path. Since each column of $\bar{\mathbf{R}}$ consists of only two nonzero elements, $(+1)$ and (-1), the above summation eliminates all terms except for the first and last ones (telescoping property of the sum). A similar proposition was represented by *Kennington and Helgason* [1980, p. 50]. Equation (A3) is modified to account for all demand nodes as follows:

$$\left(\frac{R_s}{\mathbf{R}}\right)\mathbf{P}^T = \left(\frac{\mathbf{1}}{-\mathbf{I}}\right) \qquad \mathbf{1} = [1, 1, \cdots, 1] \tag{A5}$$

Disregarding the first row of (A5) and substituting (A1) and (A2) into (A5), we have

$$(\mathbf{R}_t|\mathbf{R}_c)(\mathbf{P}_t^T/\mathbf{o}) = -\mathbf{I} \tag{A6}$$

$$\mathbf{R}_t\mathbf{P}_t^T = -\mathbf{I} \tag{A7}$$

Since $\mathbf{R}_t$ is a singular matrix [*Kennington and Helgason*, 1980, p. 58] $\mathbf{R}_t^{-1}$ exists and so the relationship between the path and the incidence matrices is given by

$$\mathbf{P}_t = -(\mathbf{R}_t^{-1})^T \tag{A8}$$

Next, it is shown how the original head constraints (equations (15) and (16)) are converted into the new from (equation (44)) following a series of elementary matrix operations.

1. Conversion of the head path constraints: the original version of the path constraints (equation (16)) is equivalent to

$$\mathbf{P\Delta h} \leq \mathbf{h}_s - \mathbf{h}_{\min} \qquad \mathbf{h}_s = [h_s, h_s, \cdots, h_s] \tag{A9}$$

Introducing the slack variables $\mathbf{h}^+$, (A9) becomes

$$\mathbf{P\Delta h} + \mathbf{h}^+ = \mathbf{h}_s - \mathbf{h}_{\min} \tag{A10}$$

dividing the hydraulic headlosses into $\mathbf{\Delta h}_t$ (for branches) and $\mathbf{\Delta h}_c$ (for chords) and using only the paths on the spanning tree,

$$\mathbf{P}_t\mathbf{\Delta h}_t + \mathbf{h}^+ = \mathbf{h}_s - \mathbf{h}_{\min} \tag{A11}$$

Substituting (A8),

$$-(\mathbf{R}_t^{-1})^T\mathbf{\Delta h}_t + \mathbf{h}^+ = \mathbf{h}_s - \mathbf{h}_{\min} \tag{A12}$$

or

$$\mathbf{R}_t^T\mathbf{h}^+ - \mathbf{\Delta h}_t = \mathbf{R}_t^T(\mathbf{h}_s - \mathbf{h}_{\min}) \tag{A13}$$

2. Conversion of the head loop constraints: the original version of the loop constraints (equation (15)) is

$$\mathbf{L\Delta h} = \mathbf{o} \tag{A14}$$

Taking the loop matrix for the set of basic loops as defined by the spanning tree T it is shown [*Deo*, 1974, p. 214] that

$$\mathbf{L} = (\mathbf{L}_t|\mathbf{I}) \tag{A15}$$

where $\mathbf{L}_t$ is a submatrix of $\mathbf{L}$ (size $(ne - nn + 1) \times (nn - 1)$) corresponding to the arcs of T and $\mathbf{I}$ is a unit matrix (size $(ne - nn + 1) \times (ne - nn + 1)$) corresponding to the set of chords. The original loop constraints are therefore

$$(\mathbf{L}_t|\mathbf{I})(\mathbf{\Delta h}_t/\mathbf{\Delta h}_c) = \mathbf{o} \tag{A16}$$

$$\mathbf{L}_t\mathbf{\Delta h}_t + \mathbf{\Delta h}_c = \mathbf{o} \tag{A17}$$

A well-known result from graph theory [*Deo*, 1974, p. 217] is that $\mathbf{RL}^T = \mathbf{o}$; thus

$$(\mathbf{R}_t|\mathbf{R}_c)(\mathbf{L}_t^T/\mathbf{I}) = \mathbf{R}_t\mathbf{L}_t^T + \mathbf{R}_c = \mathbf{o} \tag{A18}$$

or

$$\mathbf{L}_t = -(\mathbf{R}_t^{-1}\mathbf{R}_c)^T \tag{A19}$$

Substituting (A19) into (A17) results in

$$-(\mathbf{R}_t^{-1}\mathbf{R}_c)^T\Delta\mathbf{h}_t + \Delta\mathbf{h}_c = 0 \tag{A20}$$

$$-\mathbf{R}_c^T\{(\mathbf{R}_t^{-1})^T\Delta\mathbf{h}_t - \mathbf{h}_{\min}\} + \Delta\mathbf{h}_c = \mathbf{R}_c^T\mathbf{h}_{\min} \tag{A21}$$

The term in brackets equals to $\mathbf{h}^+ - \mathbf{h}_s$ (A12) and so

$$-\mathbf{R}_c^T(\mathbf{h}^+ - \mathbf{h}_s) + \Delta\mathbf{h}_c = \mathbf{R}_c^T\mathbf{h}_{\min} \tag{A22}$$

$$\mathbf{R}_c^T\mathbf{h}^+ - \Delta\mathbf{h}_c = \mathbf{R}_c^T(\mathbf{h}_s - \mathbf{h}_{\min}) \tag{A23}$$

3. Finally, combining the resulting expressions of the path and loop constraints (equations (A13) and (A23), we have

$$\left(\frac{\mathbf{R}_t^T\mathbf{h}^+ - \Delta\mathbf{h}_t}{\mathbf{R}_c^T\mathbf{h}^+ - \Delta\mathbf{h}_c}\right) = \left(\frac{\mathbf{R}_t^T(\mathbf{h}_s - \mathbf{h}_{\min})}{\mathbf{R}_c^T(\mathbf{h}_s - \mathbf{h}_{\min})}\right) \tag{A24}$$

or

$$\mathbf{R}^T\mathbf{h}^+ - \Delta\mathbf{h} = \mathbf{R}^T(\mathbf{h}_s - \mathbf{h}_{\min}) = \mathbf{R}^T\Delta\mathbf{h}_p \tag{A25}$$

which is the same as the proposed new version of the head constraints (equation (44)). The same proof holds for a multisource network. To see this an artificial node is added to the network and is connected by artificial arcs to all sources. The desired spanning tree includes the artificial arcs and is rooted at the artificial node.

APPENDIX B: EVALUATION OF THE GOF'S FIRST TERM

The first term of the GOF, which appears in (51), $\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)/\partial\mathbf{q}_l$, is the gradient of the objective function, where the optimal nonbasic variables are kept constant. That is, the optimal change in the objective function with a fixed basis. Following are the mathematical evaluation and the hydraulic interpretation of this term.

Denote the basis of $\mathbf{x}^*$ by the subscript b; then

$$\frac{\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{q}_l} = \frac{\partial\{\mathbf{c}_b^T\mathbf{x}_b^*(\mathbf{q}_l, \mathbf{x}_n^*)\}}{\partial\mathbf{q}_l} = \mathbf{c}_b^T\frac{\partial\mathbf{x}_b^*(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{q}_l} \tag{B1}$$

Expressing $\mathbf{c}_b$ by means of the optimal dual variables $\boldsymbol{\pi}$, (B1) becomes

$$\frac{\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{q}_l} = \boldsymbol{\pi}^T\mathbf{A}_b\frac{\partial\mathbf{x}_b^*(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{q}_l} \tag{B2}$$

where $\mathbf{A}_b$ is the basis of the constraint matrix in (48). The right-hand side vector of P4, $\mathbf{b}$, is given by

$$\mathbf{b} = \mathbf{A}_b(\mathbf{q}_l)\mathbf{x}_b^* + \mathbf{A}_n(\mathbf{q}_l)\mathbf{x}_n^* = \mathbf{b}(\mathbf{q}_l, \mathbf{x}_n^*, \mathbf{x}_b^*) \tag{B3}$$

Since $\mathbf{b}$ is a constant value it follows that

$$d\mathbf{b}(\mathbf{q}_l, \mathbf{x}_n^*, \mathbf{x}_b^*) = \frac{\partial\mathbf{b}}{\partial\mathbf{q}_l}d\mathbf{q}_l + \frac{\partial\mathbf{b}}{\partial\mathbf{x}_n^*}d\mathbf{x}_n^* + \frac{\partial\mathbf{b}}{\partial\mathbf{x}_b^*}d\mathbf{x}_b^* = \mathbf{o} \tag{B4}$$

Keeping $\mathbf{x}_n^*$ at constant value then

$$\frac{\partial\mathbf{b}}{\partial\mathbf{q}_l} = -\frac{\partial\mathbf{b}}{\partial\mathbf{x}_b^*}\frac{d\mathbf{x}_b^*}{d\mathbf{q}_l} = -\mathbf{A}_b\frac{\partial\mathbf{x}_b^*}{\partial\mathbf{q}_l} \tag{B5}$$

Substituting (B5) into (B2), the desired derivative is given by

$$\frac{\partial\mathcal{F}(\mathbf{q}_l, \mathbf{x}_n^*)}{\partial\mathbf{q}_l} = -\boldsymbol{\pi}^T\frac{\partial\mathbf{b}(\mathbf{q}_l, \mathbf{x}_n^*, \mathbf{x}_b^*)}{\partial\mathbf{q}_l} \tag{B6}$$

Notice that $\partial\mathbf{b}/\partial\mathbf{q}_l = \mathbf{o}$ for the length constraints (equation (17)), so that (B6) includes only the head constraints. An hydraulic interpretation of the above result may be described as follows.

1. Given an optimal solution of the pipeline diameters for some feasible flow distribution, $\mathbf{q}_l$.

2. The circular flows are changed, while keeping the pipeline diameters unchanged. The head constraints $\mathbf{b}$ are violated by the amount $\partial\mathbf{b}/\partial\mathbf{q}_l$.

3. The violated head constraints are compensated by an optimal change in the pipeline diameters (i.e., by changing the lengths of the segments $\mathbf{x}$) while keeping the flow distribution unchanged. Such a compensation is given by $\partial\text{cost}/\partial\mathbf{b} = \boldsymbol{\pi}^T$, which is the optimal change in the objective function following an incremental change in $\mathbf{b}$. The minus sign in (B6) comes to oppose the previous violation in step (2).

The above expression of $\partial\mathcal{F}/\partial\mathbf{q}_l$ corresponds to the general LPG model (P4). With regard to the particular case of the original formulation of the head constraints (equations 15) and (16)), the GOF expression can be further analyzed as follows.

The original head constraints (equations (15) and (16)) are combined such that

$$(\mathbf{L}/\mathbf{P})\bar{\mathbf{I}}\mathbf{J}(\mathbf{q}_l)\mathbf{x} \leq \mathbf{b} \tag{B7}$$

where $\mathbf{b}$ is taken with regards to the head constraints only.

Since the dual variables of the nonbinding constraints are equal to zero, (B6) becomes

$$\frac{\partial\mathcal{F}}{\partial\mathbf{q}_l} = -\boldsymbol{\pi}^T\left(\frac{\mathbf{L}}{\mathbf{P}}\right)\bar{\mathbf{I}}\frac{\partial(\mathbf{J}(\mathbf{q}_l)\mathbf{x})}{\partial\mathbf{q}_l} = -\boldsymbol{\pi}^T\left(\frac{\mathbf{L}}{\mathbf{P}}\right)\frac{\partial\Delta\mathbf{h}}{\partial\mathbf{q}_l}$$
$$= -\boldsymbol{\pi}^T\left(\frac{\mathbf{L}}{\mathbf{P}}\right)\frac{\partial\Delta\mathbf{h}}{\partial\mathbf{q}}\frac{d\mathbf{q}}{d\mathbf{q}_l} \tag{B8}$$

where $\Delta\mathbf{h}$ is the vector of the head losses in each arc and $\partial\Delta\mathbf{h}/\partial\mathbf{q}_l$ is taken for constant value of $\mathbf{x}$.

The functional relationship between the arc flows $\mathbf{q}$ and the citcular flow $\mathbf{q}_l$ is given by

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{L}^T\mathbf{q}_l \tag{B9}$$

where $\mathbf{q}_o$ is some feasible flow distribution.

Differentiating (B9) with respect to $\mathbf{q}_l$ we have

$$d\mathbf{q}/d\mathbf{q}_l = \mathbf{L}^T \tag{B10}$$

Substituting (B10) into (B8) the final term is obtained:

$$\frac{\partial\mathcal{F}}{\partial\mathbf{q}_l} = -\boldsymbol{\pi}^T\left(\frac{\mathbf{L}}{\mathbf{P}}\right)\frac{\partial\Delta\mathbf{h}}{\partial\mathbf{q}}\mathbf{L}^T = -\boldsymbol{\pi}^T\left(\frac{\mathbf{L}}{\mathbf{P}}\right)\mathbf{S}\mathbf{L}^T \tag{B11}$$

where $\mathbf{S}$ is the "resistance matrix" defined by (23). Notice that the final result is equal to the GOF as proposed by *Alperovits and Shamir* [1977], given by (25).

Following the same procedure for the alternative model, P3, the head constraints (equation (44)) are given by

$$\mathbf{R}^T\mathbf{h}^+ - \Delta\mathbf{h}(\mathbf{q}_l) = \mathbf{b} \tag{B12}$$

and the resulting expression for $\partial\mathcal{F}/\partial\mathbf{q}_l$ is

$$\partial\mathcal{F}/\partial \mathbf{q}_l = \boldsymbol{\pi}^T \mathbf{S}\mathbf{L}^T \quad \text{(B13)}$$

APPENDIX C: THE PROJECTED GOF ALGORITHM

Following are a (1) brief description of the GOF projection algorithm and (2) a solved example of the simple network, shown in Figure 1. Theoretical background and a detailed description of the gradient projection method is given by *Reklaitis et al.* [1983, pp. 402–419].

According to the LPG second stage, the cost function is minimized in the space of the circular flows only. The corresponding constraints are the assumed flow directions and the minimum flow discharge in each arc. The problem may thus be formulated as follows:

$$\min \{\text{cost}(\mathbf{q}_l) | \mathbf{L}^T\mathbf{q}_l \geq \mathbf{b}\} \quad \text{(C1)}$$

where $\mathbf{b} = \mathbf{q}_{\min} - \mathbf{q}_0$. Given a feasible flow distribution $\mathbf{q}_l^{(t)}$ and a convergence parameter $\varepsilon \geq 0$, the algorithm steps are as follows:

Step 1. Solve the LPG first stage, P2 or P3, and calculate the GOF by (53) for P2 or (54) for P3.

Step 2. Identify the active set of constraints for which

$$\bar{\mathbf{L}}^T\mathbf{q}_l^{(t)} = \bar{\mathbf{b}} \qquad \text{or} \qquad L_j\mathbf{q}_l^{(t)} = b_j \qquad j = (1, \cdots, J) \quad \text{(C2)}$$

where L_j is a column of $\bar{\mathbf{L}}$ and the bar denotes the matrix and vector corresponding to the active constraints.

Step 3. Calculate the projected GOF by

$$\boldsymbol{\omega}^{(t)} = (\mathbf{I} - \bar{\mathbf{L}}(\bar{\mathbf{L}}^T\bar{\mathbf{L}})^{-1}\bar{\mathbf{L}}^T)(-\text{GOF}) \quad \text{(C3)}$$

Step 4. If $\|\boldsymbol{\omega}^{(t)}\| \leq \varepsilon$, go to step 7, otherwise, determine the maximum step size

$$\alpha_{\max} = \min\left\{\max\left[0, \frac{b_k - L_k\mathbf{q}_l^{(t)}}{L_k\boldsymbol{\omega}^{(t)}} \quad \text{or} \quad \infty \text{ if } L_k\omega^{(t)} \geq 0\right] \quad k = 1, \cdots, ne\right\} \quad \text{(C4)}$$

Step 5. Solve the line search problem for α^*:

$$\min\{\text{cost}(\mathbf{q}_l^{(t)} + \alpha\boldsymbol{\omega}^{(t)})\} \qquad 0 \leq \alpha \leq \alpha_{\max} \quad \text{(C5)}$$

Step 6. Set $\mathbf{q}_l^{(t+1)} = \mathbf{q}_l^{(t)} + \alpha^*\boldsymbol{\omega}^{(t)}$ and return to step 1.

Step 7. Test optimality conditions according to

$$(\bar{\mathbf{L}}^T\bar{\mathbf{L}})^{-1}\bar{\mathbf{L}}^T(\text{GOF}) \quad \text{(C6)}$$

If the terms are all nonnegative, terminate. Otherwise, the constraint corresponding to the most negative term is deleted from the active constraint set and step 2 is reinitiated.

Due to the concave nature of the objective function, cost($\mathbf{q}_l$), it is usually found that step 5 and step 7 are not necessary. That is, instead of line search in step 5, $\alpha_{\max}$ is used to set the new flow distribution, while step 7 is always satisfied.

An implication of the above algorithm is presented with regard to the simple network shown in Figure 1. Given a minimum flow discharge of 10 m^3/hour and the initial flow distribution in Figure 1. The flow constraints corresponding to the second stage (equation (59)) are

$$\begin{matrix}1:\\2:\\3:\\4:\\5:\\6:\\7:\\8:\end{matrix}\begin{pmatrix}0 & 0\\-1 & 0\\1 & 0\\1 & -1\\0 & 1\\0 & 1\\-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}q_{l1}\\q_{l2}\end{pmatrix} \geq \begin{pmatrix}-1110\\-360\\-640\\10\\-520\\-190\\-260\\10\end{pmatrix}$$

Starting from point A in Figure 2, the first active constraint is met at point B where q_{l1} = 63.7 m^3/hour and q_{l2} = 10.0 m^3/hour. Consider point B to be a new starting point. The algorithm proceeds as follows.

Step 1. P2 is solved at point B and its corresponding GOF is calculated by (53):

$$\text{GOF} = (0.103, 0.439)^T$$

Step 2. Constraint (8) becomes active, that is,

$$0 \times 63.7 + 1 \times 10 = 10$$

and so $\bar{\mathbf{L}}^T = [0, 1]$.

Step 3. The projected GOF is given by

$$\boldsymbol{\omega} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\end{bmatrix}\right\}\begin{pmatrix}-0.103\\-0.439\end{pmatrix} = \begin{pmatrix}-0.103\\0\end{pmatrix}$$

Step 4. The maximum step size is

$$\alpha_{\max} = \min\left\{\begin{matrix}1\text{: max }\{0, \quad \infty\ \}\\2\text{: max }\{0, \quad \infty\ \}\\3\text{: max }\left\{0, \dfrac{-640-63.7}{-0.103}\right\}\\4\text{: max }\left\{0, \dfrac{10-63.7+10}{-0.103}\right\}\\5\text{: max }\{0, \quad \infty\ \}\\6\text{: max }\{0, \quad \infty\ \}\\7\text{: max }\{0, \quad \infty\ \}\\8\text{: max }\{0, \quad \infty\ \}\end{matrix}\right\} = 424.27$$

Step 5. The maximum step size is adopted:

$$\alpha^* = \alpha_{\max} = 424.27$$

Step 6. The new flow distribution is

$$\mathbf{q}_l = \begin{pmatrix}63.7\\10\end{pmatrix} + 424.27\begin{pmatrix}-0.103\\0\end{pmatrix} = \begin{pmatrix}20\\10\end{pmatrix}$$

which corresponds to point C in Figure 2. The next iteration proceeds as follows.

Step 1. P2 is solved for point C:

$$\text{cost} = 417{,}500 \text{ units} \qquad \text{GOF} = [0.183, 0.267]^T$$

Step 2. Constraints (4) and (8) are active and so

$$\bar{\mathbf{L}}^T = \begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} \qquad (\bar{\mathbf{L}}^T\bar{\mathbf{L}})^{-1} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$$

Step 3. The projected gradient is

$$\boldsymbol{\omega} = \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\right\}\begin{pmatrix}-0.183\\-0.267\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Step 4. $\|\omega\| < \varepsilon$

Step 7. Optimality critera are tetsted:

$$\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0.183\\ 0.267\end{pmatrix} = \begin{pmatrix}0.183\\ 0.450\end{pmatrix} > 0 \Rightarrow \text{O.K.}$$

Therefore point C satisfies the Kuhn-Tucker conditions with optimal cost of 417,500 units.

NOTATION

$\mathbf{a} = [a_e]$	vector of arc length.
$\mathbf{A}$	general constraint matrix.
$\mathbf{A}_b$	basis of the constraint matrix.
$\mathbf{b}$	right-hand side of a general constraint set.
$\mathbf{c} = [c_{e,d}]$	cost per unit length in segment $x_{e,d}$.
chw	Hazen-Williams coefficient.
cost	cost of pipeline network.
$d \in D_e$	pipe diameter, belongs to the commercial diameter set of arc e.
$e \in E$	an arc which belongs to the arc set of the network.
$\mathbf{e}$	unit vector.
$\mathbf{E}$	elementary matrix.
GOF	gradient of the objective function.
$\mathbf{h} = [h_i]$	hydraulic head in each node.
$\mathbf{h}^+$	surplus/slack head variables.
$\mathbf{h}_s$	vector consists of identical elements h_s.
$\mathbf{I}$	unit (identity) matrix.
$\bar{\mathbf{I}}$	matrix of the internal arrangement of the pipe segments within the arc set.
$\mathbf{J} = [j_{e,d}]$	hydraulic gradient in the pipe segment $x_{e,d}$.
$l \in L$	loop, belongs to the set of basic loops.
$\mathbf{L}$	loop matrix.
$\bar{\mathbf{L}}$	submatrix of $\mathbf{L}$.
nd	number of commercial pipe diameters allocated for each arc.
ne	number of arcs in the network.
nl	number of basic loops.
nn	number of nodes in the network.
np	number of critical paths.
ns	number of pipe segments.
$p \in P$	critical path belongs to the critical path set.
$\mathbf{P}$	path matrix.
$\mathbf{q} = [q_e]$	flow discharge in each arc.
$\mathbf{q}_0$	a given feasible flow distribution.
$\mathbf{q}_l = [q_{l(i)}]$	vector of circular flows.
$\mathbf{q}_{\min}$	minimum allowed flow in each arc.
$\bar{\mathbf{R}}$	incidence matrix.
$\mathbf{R}$	reduced incidence matrix.
$\mathbf{S}$	"resistance matrix" of the pipeline network.
T	spanning tree.
$\mathbf{x} = [x_{e,d}]$	length of pipe segment with diameter d in arc e.
$\Delta h_{e,d}$	head loss along the pipe segment $x_{e,d}$.
$\mathbf{\Delta h}_p$	maximum allowed headloss in each path.
$\boldsymbol{\omega}$	projected GOF.
π	dual variable corresponding to the path and loop constraints.

Subscripts

b	basic.
c	chord.
d	diameter.
e	arc.
l	loop.
n	nonbasic.
o	old.
p	path.
s	source.
t	tree.

Superscripts

T	transpose.
$*$	optimal.

Acknowledgments. This work was conducted at the Technion, as part of a project at MIT, under grant CEE 8408369 of the National Science Foundation. Partial support was given by the VPR Fund at the Technion, Project 013-0067.

REFERENCES

Alperovits, A., and U. Shamir, Design of optimal water distribution systems, *Water Resour. Res.*, *13*(6), 885–900, 1977.

Deo, E., Graph theory with application to engineering and computer science, Prentice-Hall, Englewood Cliffs, N. J., 1974.

Fujiwara, O., B. Jenchaimahakoon, and N. C. P. Edirisinghe, A modified linear programming gradient method for optimal design of looped water distribution networks, *Water Resour. Res.*, *23*(6), 977–982, 1987.

Goulter, I. C., B. M. Lussier, and D. R. Morgan, Implications of head loss path choice in the optimization of water distribution network, *Water Resour. Res.*, *22*(5), 819–822, 1986.

Hadley, G., *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, Mass., 1964.

Kennington, J. L., and R. V. Helgason, Algorithms for networks programming, John Wiley and Sons, New York, 1980.

Morgan, D. R., and I. C. Goulter, Optimal urban water distribution design, *Water Resour. Res.*, *21*(5), 642–652, 1985.

Orth, H. M., *Model-Based Design of Water Distribution and Sewage Systems*, Wiley-Interscience, New York, 1986.

Quindry, G. E., E. D. Brill, J. C. Liberman, and A. R. Robinson, Comment on "Design of optimal water distribution system" by E. Alperovits and U. Shamir, *Water Resour. Res.*, *15*(6), 1651–1654, 1979.

Quindry, G. E., E. D. Brill, and J. C. Liebman, Optimization of looped water distribution systems, *J. Environ. Eng. Div. Am. Soc. Civ. Eng.*, *107*(EE4), 665–667, 1981.

Reklaitis, G. V., A Ravindran and K. M. Ragsdell, *Engineering Optimization—Methods and Application*, John Wiley and Sons, New York, 1983.

Rosen, J. B., The gradient projection method for non-linear programming, 1, Linear constraints, *SIAM J. Appl. Math.*, *8*, 181–217, 1960.

Saphir, Y. N., Optimization of water distribution systems (in Hebrew), 159 pp., M.S. thesis, Technion, Haifa, Israel, April 1983.

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(Received October 10, 1988;
revised January 17, 1989;
accepted February 13, 1989.)