Design of Optimal Water Distribution Systems

E. ALPEROVITS[1] AND U. SHAMIR

Faculty of Civil Engineering, Technion-lsrael Institute of Technology, Haifa, Israel

A method called linear programing gradient (LPG) is presented, by which the optimal design of a water distribution system can be obtained. The system is a pipeline network, which delivers known demands from sources to consumers and may contain pumps, valves, and reservoirs. Operation of the system under each of a set of demand loadings is considered explicitly in the optimization. The decision variables thus include design parameters, i.e.. pipe diameters, pump capacities and reservoir elevations, and operational parameters, i.e., the pumps to be operated and the valve settings for each of the loading conditions. The objective function, to be minimized, reflects the overall cost capital plus present value of operating costs. The constraints are that demands are to be met and pressures at selected nodes in the network are to be within specilied limits. The solution is obtained via a hierarchial decomposition of the optimization problem. The primary variables are the flows in the network. For each flow distribution the other decision variables arc optimized by linear programing. Postoptimality analysis of the linear program provides the information necessary to compute the gradient of the total cost with respect to changes in the How distribution. The gradient is used to change the flows so that a (local) optimum is approached. The method was implemented in a computer program. Solved examples are presented.

INTRODUCTION

Water distribution systems connect consumers to sources of water, using hydraulic components, such as pipes, valves, and reservoirs. The engineer faced with the design of such a system, or of additions to an existing system, has to seiect the sizes of its components. Also he has to consider the way in which the operational components, pumps and valves, will be used to supply the required demands with adequate pressures. The network has to perform adequately under varying demand loads, and in the design process, one considers several significant loads: maximum hourly, average daily, low-demand periods during which reservoirs are to be filled, etc. Operational decisions for these loads are essentially part of the design process, since one cannot separate the so-called design decisions, i.e.. the sizing of components, from the operational decisions; they are two inseparable parts of one problem.

This paper presents a method for optimizing the design of a water distribution system: sizing its components and setting the operational decisions for pumps and valves under a number of loading conditions, those which are considered 'typical' or 'critical.' The detailed sequence of operation of the system, say. over a day. cannot be determined by this method. Still, inclusion in the design process of the operational decisions for the typical and critical loadings insures that the resulting design properly reflects the operation. Also the method can be used to determine optimal operating rules for an existing system.

Work *on* optimal design and operation of water distribution systems up to 1973 has been reviewed by one of the authors [*Shamir,* 1973, 1974]. Subsequent works in this area are those by *Watanatada* [1973], *Hamberg* [1974], and *Rasmusen* [1976]. Methods for optimal design of looped systems can be separated into two categories: (1) methods which require the use of a network solver (at each iteration of the optimization, one first solves for the heads and flows in the network, then uses this solution in some procedure to modify the design [*Jacoby,* 1968; *Kally,* 1972; *Watanatada,* 1973; ***Shamir,*** 1974; *Rasmusen,* 1976]) and (2) methods which do not use a conventional network solver. *Lai and Schaake* [1969] and *Kohlhaas and Mattern* [1971] did not use a network solver, but both works treated the case in which the head distribution in the network is fixed. To the best of our knowledge, the linear programing gradient (LPG) method presented in this paper is the first to incorporate the flow solution into the optimization procedure, without making any assumptions about the hydraulic solution of the network. We believe that this is not merely a technical detail, since, as will be demonstrated in the paper, it enables optimization for multiple loadings and explicit inclusion of operational decisions.

The next section presents a method for designing branching networks by linear programing (LP), which is a basic component in the LPG method. Then the basic LPG method will be developed for a pipeline network operating by gravity for one loading condition. A simple example will complete this presentation. Next, the method will be extended to real networks, which contain pumps, valves, and reservoirs and which operate under multiple loadings. An additional example will demonstrate the application of the full method.

OPTIMAL DLSIGN OF BRANCHING NETWORKS BY LP

Consider a branching network supplied from a number of sources by gravity. At each of the nodes of the network, $j = 1, \ldots, N'$, a given demand d_j has to be satisfied. The head of each node H_j is to be between a given minimum $H_{\min j}$ and a given maximum $H_{\max j}$. The layout of the network is given, and the length of the link (pipeline) connecting nodes (andjf is'Ly. The LP design procedure [*Karmeli et al.,* 1968: *Gupta,* 1969; *Gupta et al.,* 1972: *Hamberg,* 1974] is based on a special selection of the decision variables: instead o(selecting pipe diameters, allow a set of'candidate diameters' in each link, the decision variables being the lengths of the segments of constant diameter within the link. Denote by x_{ijm} the length of the pipe segment of the *mth* diameter in the link connecting nodes /and /: then

$$\sum_m x_{ijm} = L_{ij} \tag{1}$$

has to hold for all links, where the group of candidate diameters may be different for each link. In a branching network, once the demands d_j are known, the discharge in each link Q_{ij} can easily be computed. The head loss in segment *in* of this link is

[1]'Now at Tahal. Tel-Aviv, Israel.

Paper number 7W0382,

$$\Delta H_{ijm} = J_{ijm} x_{ijm} \quad (2)$$

where *J,* the hydraulic gradient, depends on the pipe properties, diameter and roughness, and on the discharge. Specifically, by using the Hazen-Williams equation,

$$J = \alpha (Q/C)^{1.852} D^{-4.87} \quad (3)$$

where *Q* is the discharge, *C* is the Hazen-Williams coefficient, *D* is the pipe diameter, and « is a coefficient, whose value depends on the units used (for example, for *Q* in mVh and *D* in centimeters, *a* = 1.526 X 10[1]"; for *Q* in ftVs and *D* in inches, *a* = 8.515 X 10^5).

Starting from any node in the system $ at which the head is known in advance (for example, at a reservoir' or at some source), one may write for node *n,*

$$H \min_n \le H_s \pm \sum_{i,j} \sum_m J_{ijm} x_{ijm} \le H \max_n \quad (4)$$

where the first summation is over all links (/, *j)* connecting node *s* with node *n,* and the second summation is over all segments *m* in each link. The sign of the terms depends on the direction of flow. Equation (4) represents two linear constraints. The *H* min constraint usually results from service performance requirements. The *H* max constraint may result From service performance requirements, or from technological limitation on the pressure-bearing capacity of the pipes.

The cost of a pipeline is assumed to be linearly proportional to its length, a reasonable assumption under most circumstances. Without undue complication of the linear formulation the cost may be a function of location, i.e., the link. Thus the total cost of the pipeline network is

$$\sum_{i,j} \sum_m c_{ijm} x_{ijm} \quad (5)$$

Minimization of (5) subject to constraints of the form (4) and to nonnegativity requirements

$$x_{ijm} \ge 0 \quad (6)$$

is a linear program.

The objective function can be expanded to account for the cost of pumps and their operation over time *[Karmeli et al.,* 1968] by using linear cost functions. Nonlinear cost functions for pumping costs will be dealt with later in this paper.

It should be noted that preselection of the candidate diameters for each link introduces an implicit constraint into the optimization problem, by virtue *of* the fact that the range of possible diameters has been limited. Restriction of the number of possible diameters may be based on some constraint from engineering practice; for example, only certain diameters may be commercially available. Usually, however, limiting the number of diameters in the candidate list is aimed at reducing the number of decision variables and the computational effort and does not reflect a real constraint. When this is done, the implicit constraint introduced by restricting the diameters in the lists for the links may be binding at the computed optimum, and a true optimal solution may not be reached. At the optimal solution, no link should be made entirely of a diameter at one extreme of its list of candidate diameters. I f this does happen, the list of candidate diameters for this link should be expanded in the proper direction, and the problem solved again, until this constraint is not binding for all links. It can be shown that at the optimum, each link will contain at most two segments, their diameters being adjacent on the candidate list for that link.

The optimal solution should be examined for segments whose optimal length is too small to be of practical significance, and they can be eliminated. Although the resulting design is not strictly optimal and possibly even does not exactly satisfy constraints (4), it is probably acceptable. If it is not, slight modifications may be needed. In engineering practice it has been the custom to select a single diameter for the entire length of each link. If this is done, the design will not be optimal.

To keep computation time down, one should attempt to reduce the number of constraints (which is the prime computational factor in a linear program; the number of variables is less important). Constraints of type (4) should be written only for selected nodes in the network. One may start with few such constraints and examine the solution. If it satisfies all head constraints at the other nodes, the solution is acceptable. Otherwise, one adds constraints for those nodes at which they were not satisfied and solves again.

When storage reservoirs are to be designed by using a linear program, their cost has to be approximated by a linear function of the water level in the reservoir. The reservoir is considered a source with a fixed head.

M ore than one set of demands can be handled by the same formulation. Each loading adds an additional set of constraints to the LP problem; the entire set is then solved simultaneously. If energy costs are included, the objective function contains a weighted sum of the energy costs of operating under the different loadings.

BASIC LPG METHOD

The LPG method deals with looped networks and decomposes the optimization problem into a hierarchy of two levels as depicted in Figure 1. We shall present the method for a pipeline network operating under gravity for one loading. Later sections will extend the basic method to cover multiple loadings and to allow for pumps, valves, and reservoirs.

The first step in developing the LPG method is to consider optimization of the design when the distribution of flows'in the network is assumed to be known. We adopt the formulation given by equations (1), (2), (4), and (5), in which the lengths of

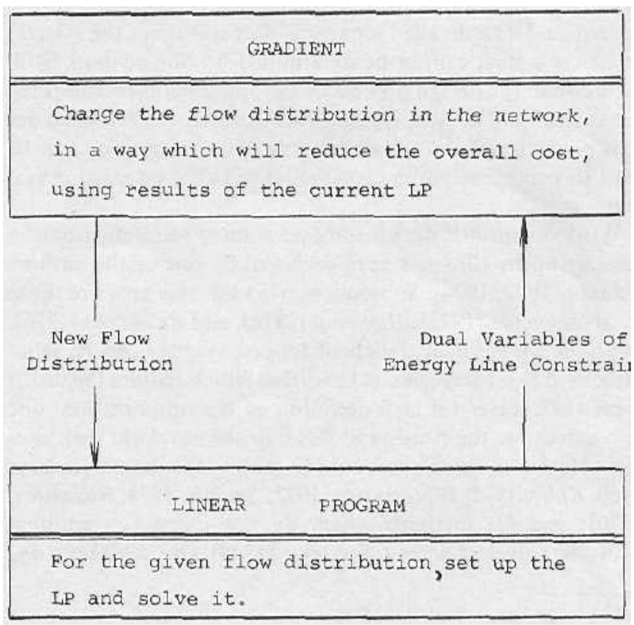


Fig. 1. Overview of the LPG method.

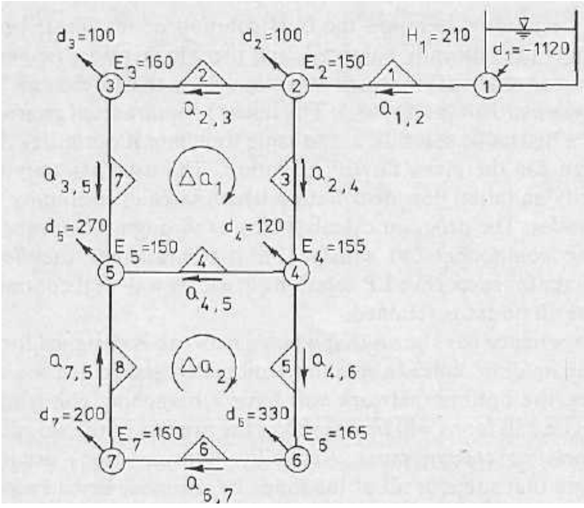


Fig. 2. A two-loop network supplied by gravity.

the segments of constant diameter in each link are the decision variables. If the flows in the links are assumed to be known (how they change will be explained later), then the constraints on heads at nodes, equation (4), can readily be formulated. Constraints (I), (4), and (6) do not. however, suffice to ensure a feasible solution, and one has to add the conditions that the head losses along certain paths in the network satisfy the following type of constraint:

$$\sum_{i,j} \sum_{m} J_{ijm} x_{ijm} = b_p \tag{7}$$

where b_p is the known head difference between the end nodes of the path *p*. The first summation is taken over all links *i,j* in the path, and the second over all segments in the link. Equation (7) has to hold for all closed paths, i.e.. loops, with 6 = 0. For each pair of nodes at which the heads are fixed, equation (7) is formulated by proceeding along any path which connects the two nodes, starting at the node with the higher head, so that $b_p > 0$. These constraints having been added, the linear program

TABLE l<j. Basic Design Data for the Network in Figure 2 With Two Loops, One Source, and a Single Load

	Value
Number of sections	8
Number of nodes	7
Greatest diameter allowed, in.	24
Maximum capacity of pumps, hp	0
Number of different flow distributions	1
Number of pumps	0
Number of valves	0
Number of dummy variables	0
Number of storages	0
Lowest allowable gradient, promilles	0.5
Highest allowable gradient, promilles	50.0
Number of loop/source equations in which flow is allowed to change	2
Initial step size, m³/h	5.0
Number of free loops	2

The computation stopped when max *DQ(I)* equaled 1.0 or after 50 major iterations; Results will be printed out every five major iterations or whenever best total cost is further improved. Flow change in loop / will be executed only if *DQ{I)/DQ* max is greater than 0.20. The number **of** minor iterations allowed after a feasible solution has been reached for a flow distribution was 20. The local solution is considered to be reached if last improvement per iteration equals 0.0%.

with the objective function (5) can be solved, and the set of optimal segments will be such that the network is hydraulically balanced. If we denote by Q a vector of Hows in all links, which satisfy continuity at all nodes, then for any Q the optima! cost of the network may be written as

$$\text{cost} = \text{LP(Q)} \tag{8}$$

where LP simply denotes thai cost is the outcome of a linear program. The comments which were made in a previous section about making certain that the selection of the candidate diameters does not impose an unwanted constraint on the solution apply here too,

The next stage is to develop a method for systematically changing Q with the aim of improving cost, since now we have the relation (8) which ensures that for each Q the best cost can be found. The flow distribution thus becomes the primary decision variable, and the actual design variables result from the LP solution. The method for changing Q is based on use of the dual variables of the constraints (7), which aid in defining a gradient move. AQ, a vector of changes in the flows in all links, is sought such that LP(Q + AQ) < LP(Q), and the 'move' has been in the best possible direction, i.e., along the negative gradient of cost. If one denotes by AQ_n the change in flow in path /?, then

$$\frac{\partial(\text{cost})}{\partial(\Delta Q_p)} = \frac{\partial(\text{cost})}{\partial b_p} \cdot \frac{\partial b_p}{\partial(\Delta Q_p)} = W_p \cdot \frac{\partial b_p}{\partial(\Delta Q_p)} \tag{9}$$

Here W_p is the value of the dual variable of the constraints of type (7) for the path, which may be positive or negative, since (7) is an equality constraint. The second term on the right is computed from equation (7). The notation used in (9) is not strictly correct, because b_p is a given constant [hat does not change. Actually, b_p in (9) stands for the following expression, which does change with Q:

$$b_p = \sum_{i,j} \sum_{m} J_{ijm} x_{ijm} = \sum_{i,j} \sum_{m} [\alpha Q_{ij}^{1.852} C_{ijm}^{-1.852} D_{ijm}^{-4.87} x_{ijm}] \tag{10}$$

Then

$$\frac{\partial b_p}{\partial(\Delta Q_p)} \equiv \frac{\partial b_p}{\partial(Q_p)} = \sum_{i,j} \sum_{m} [1.852 \alpha Q_{ij}^{0.852} C_{ijm}^{-1.852} \cdot D_{ijm}^{-4.87} x_{ijm}] = \sum_{i,j} \sum_{m} [1.852(\Delta H_{ijm}/Q_{ij})] \tag{II}$$

The first identity *on* the left holds because both *8(AQ$_P$)* and *8(Q$_P$)* are incremental changes in flow in the same path. Since

Diameter	Unit Cost
1	2.0
2	5.0
3	8.0
4	11.0
6	16.0
8	23.0
10	32.0
12	50.0
14	60.0
16	90.0
18	130.0
20	170.0
22	300.0
24	550.0

TABLE \b. Basic Cost Data for Pipes

TABLE 2*a*. Node Data

Node	Elevation, m	Minimum Pressure Allowed	Consumption, m³/h	
			Load 1	Load 2
1	210.0	0.0	−1120.0	0.0
2	150.0	30.0	100.0	0.0
3	160.0	30.0	100.0	0.0
4	155.0	30.0	120.0	0.0
5	150.0	30.0	270.0	0.0
6	165.0	30.0	330.0	0.0
7	160.0	30.0	200.0	0.0

the constant 1.852 appears in all components of the gradient, and we are interested only in their relative magnitude, we can leave the constant out and write the component of the gradient vector G as

$$G_p = \frac{\partial(\text{cost})}{\partial(\Delta Q_p)} = W_p \sum_{l,j} (1/Q_{lj}) \sum_m \Delta H_{ljm} \tag{12}$$

It should be kept in mind that the summations are always performed only for the appropriate links, i.e., those belonging to the /Jth path. At each LP solution, ΔH_{ljm} and Q_{lj} have been used in setting up the linear program, and one merely needs to perform the appropriate summations and to multiply by the dual variables obtained in the solution of the linear program.

The components of the vector of changes in flows ΔQ are made proportional to the corresponding components of the gradient vector given by (12). The distance to move along this vector (the step size) has now to be determined. The changes in flows should be such that the step size is optimized, i.e., by finding β from

$$\min_{\beta} [LP(Q + \beta\Delta Q)] \tag{13}$$

No simple way was found to do this, and the heuristic approach was adopted. A step size, given in terms of a change in flow, is fixed at the start of the program (given by the user as input data). The flow component which has the largest (absolute) value of the gradient component is given a flow change of the specified step size, and the other flows are changed by quantities reduced by the ratio of the appropriate gradient components to the largest gradient component. The step is thus in the direction of the gradient, its maximum component being determined by a user-supplied value. The program also contains a routine for increasing or reducing the step size from one iteration to the next, based on the success or failure of previous steps. The overall iterative procedure stops when no improvement is achieved with the minimum step size allowed (a user-supplied parameter), or after a prescribed number of iterations has been exceeded.

At each flow iteration the final solution of the linear program is hydraulically balanced, and there is therefore no need for a conventional network solver (such as that presented by *Shamir and Howard* [1968]). The linear program itself guarantees a hydraulic solution at the same time that it optimizes the design for the given flow distribution. The user has only to specify an initial flow distribution which satisfies continuity at all nodes, The program calculates losses through the network, sizing components to satisfy (7); it then makes the flow changes for successive LP solutions in such a way that continuity at all nodes is retained.

Experience has shown that when a network is designed for a single loading, unless a minimum diameter is specified for all pipes, the optimal network will have a branching configuration; i.e.. all loops will be opened in the process of the solution by deleting certain pipes. Reliability considerations usually dictate that some or all of the loops be retained. For all pipes in these loops, one specifies a minimal (nonzero) diameter. The tendency toward a branching network still remains, and certain pipes will be at their minimum diameter. The additional cost of reliability can thus be determined as the difference between two optimal solutions, one without the minimal diameter requirement, the other with it. Forcing the network to have a fully looped configuration is not a satisfactory way of defining reliability. A more intrinsic definition is needed, one which depends on a performance criterion for specified emergency situations. More work should be done in this area.

SIMPLE EXAMPLE

Consider the network shown in Figure 2, which has eight pipes arranged in two loops and is fed by gravity from a constant head reservoir. The demands are given, and the head at each node is to be at least 30 m above the ground elevation of the node, denoted by E_l in the figure. Tables la and \b give the basic data. Costs are given in arbitrary units. (Several pipe classes of varying wall thickness and therefore of different costs and pressure-bearing capacities can be introduced, but in this problem, only one class was specified.)

The maximum diameter allowed is 24 in. Also limits are set on the minimum and maximum hydraulic gradients (Jin equation (3)) in the pipes: 0.0005 and 0.05 in the example. This means that no link can be eliminated completely, although it may be made as small as 1 in. (the lowest diameter on the list) as long as the hydraulic gradient in it does not exceed 0.05 (a very large gradient, probably several times larger than normal values in pipelines).

Tables *la* and *2b* give a summary of node and pipe data. For each pipe, there is an initial flow, selected arbitrarily but so that continuity at nodes is satisfied. Table 3 summarizes the

TABLE 2*b*. Section Data

Section	Length, m	*C*	Range of Allowable in.	Class	Initial Flow Distribution, m³/h		Selected Diameters, in.
					Load1	Load 2 '	
1	1000.0	130.0	0-24	1	1120.0	0.0	12, 14, 16, 18,20
2	1000.0	130.0	0-24	1	220.0	0.0	6, 8, 10, 12,14
3	1000.0	130.0	0-24	1	800.0	0.0	10, 12, 14. 16,18
4	1000.0	130.0	0-24	1	30.0	0.0	3, 4, 6, 8,
5	1000.0	130.0	0-24	1	650.0	0.0	10, 12, 14,16. 18
6	1000.0	130.0	0-24	i	320.0	0.0	8, 10, 12, 14. 16
7	1000.0	130.0	0-24	1	120.0	0.0	6, 8, 10, 12, 14
8	1000.0	130.0	0-24	1	120.0	0,0	6, 8, 10,12, 14

TABLE 3. Structure of the Initial Linear Program: Strings of Pipes for Pressure or Loop Constraints

Begin Node	End Node	Load	Number Order of Sections Connected Between the Nodes
			Pressure equation
1	2	1	1
1	3	1	1, 2
1	4	1	1, 3
1	5	1	1, 3, 4
1	6	1	1, 3, 5
1	7	1	1, 3, 5, 6
			Loop equation
2	2	1	3, 4, −7, −2
7	7	1	5, 6, 8, −4

1 he constraint data include *39* variables, six pressure equations, two loop equations, no equations between sources, and a coefficient matrix of 16 rows and 55 columns.

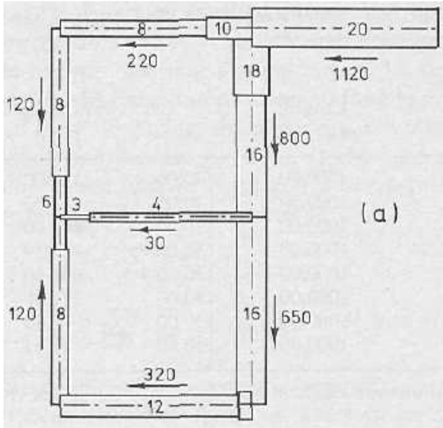
linear program which was set up initially. There are 39 variables (4-5 candidate diameters for each of the eight pipes), six pressure equations for each node except I. at which the head is fixed, and two loop equations. Because only H min is specified at each node (the maximum is unrestricted in this example), there are six head constraints (equation (4)). Their structure is listed in Table 3 by showing the 'strings* of pipes in each constraint. For example, for node 7 the constraint is formulated by going along pipes 1,3,5, and 6. The constraint is thus

$$-\sum J_{1,2,m} x_{1,2,m} - \sum J_{2,4,m} x_{2,4,m} - \sum J_{4,6,m} x_{4,6,m} - \sum J_{6,7,m} x_{6,7,m} \geq H \min_7 - H_1 \quad (14)$$

where, for example.

$$J_{1,2,m} = \alpha(Q_{1,2}/C_{1,2,m})^{1.852} D_{1,2,m}^{-4.87} \quad (15)$$

There are two loop equations, whose strings of pipes are listed also in Table 3. A negative sign means that the flow initially assumed is in a direction opposite to that taken in formulating the hydraulic head line continuity constraint (equation (7)). For example, the equation for the upper loop starts and ends at node 2 and goes along pipes 3, 4, 7. and 2. This loop equation is



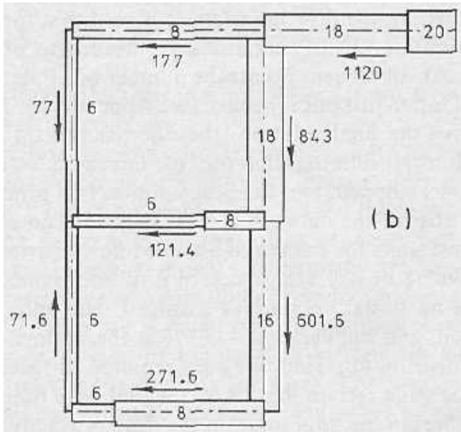


Fig. 3. *(a)* initial flow distribution and its linear programing design at a cost of 493.779. *(b)* Final flow distribution and optimal design at a cost of 479.525.

$$\sum J_{2,4,m} x_{2,4,m} + \sum J_{4,5,m} x_{4,5,m} - \sum J_{3,5,m} x_{3,5,m} - \sum J_{2,3,m} x_{2,3,m} = 0 \quad (16)$$

Table 4 summarizes the intermediate results of the computa-

TABLE 4. Intermediate Results of the Computations for the Network in Figure 2

Flow Iteration	Number of LP Iterations	Total Cost of Network	Loop 1				Loop 2			
			$\partial b_1/\partial Q_1$	W_1	G_1	ΔQ_1	$\partial b_2/\partial Q_2$	W_2	G_2	ΔQ_2
1	32	493,776	0.820	377	309	5.00	0.764	−377	−288	−4.66
2	9	493,475	0.663	408	271	5.00	0.606	−406	−246	−4.54
3	3	493,665	0.571	408	233	3.00	0.512	−406	−208	−2.68
4	2	493,313	0.521	466	243	3.00	0.473	−457	−216	−2.67
5	1	492,686	0.491	491	241	3.00	0.441	−479	−211	−2.63
6	1	491,910	0.467	519	242	3.00	0.416	−502	−209	−2.59
7	1	491,013	0.447	549	245	3.00	0.395	−526	−208	−2.54
8	1	490,015	0.432	581	251	3.00	0.378	−552	−208	−2.49
9	1	488,924	0.419	617	258	3.00	0.363	−579	−210	−2.44
10	1	487,746	0.409	656	268	3.00	0.352	−608	−214	−2.36
11	1	486,484	0.401	699	280	3.00	0.342	−638	−218	−2.33
12	1	485,226	0.397	746	296	3.00	0.333	−669	−223	−2.26
13	1	483,890	0.396	799	316	3.00	0.326	−702	−229	−2.17
14	6	483,251	0.394	184	72	0.92	0.320	−736	−236	−3.00
15	1	482,354	0.389	184	71	0.85	0.320	−788	−252	−3.00
16	1	481,359	0.384	184	70	0.78	0.320	−845	−270	−3.00
17	1	480,260	0.379	184	69	0.71	0.321	−909	−292	−3.00
18	1	479,525	0.374	184	69	3.00	0.322	−85	−27	−1.20
19	1	480,000	0.372	184	68	1.80	0.319	−85	−27	−0.70

TABLE *5a*. Final Results for the Design of the Network in Figure 2: Section Data and Optimal Diameters for Iteration 18

Section	Length, m	C	Load 1, m³/h	Load 2, m³/h	Loop 1 Diameter, in.	Loop 1 Length, m	Loop 2 Diameter, in.	Loop 2 Length, m	HF1, m
1	1000.00	130.00	1120.00	0.0	18.00	744.00	20.00	255.97	6.57
2	1000.00	130.00	177.00	0.0	8.00	996.37	6.00	3.61	12.60
3	1000.00	130.00	843.00	0.0	18.00	999.98	0.0	0.0	4.32
4	1000.00	130.00	121.39	0.0	6.00	680.62	8.00	319.38	19.11
5	1000.00	130.00	601.60	0.0	16.00	1000.00	0.0	0.0	4.11
6	1000.00	130.00	271.60	0.0	10.00	215.06	12.00	784.94	5.00
7	1000.00	130.00	77.00	0.0	6.00	999.99	0.0	0.0	10.83
8	1000.00	130.00	71.61	0.0	6.00	990.91	4.00	9.06	10.00

The solution was reached after six minor iterations. The total network cost (there were no penalty costs) was 479,525.

tions. The first column on the right gives the number of LP iterations within the flow iteration. It is seen that for the first flow distribution, 32 LP iterations are needed to obtain the optimum. At subsequent points the number of LP iterations is very small; often just one is needed (see Appendix 1). Tables *5a* and *5b* give the final solution, the one reached on the 18th iteration (on the 19th iteration the cost increased, so the computation was stopped, and the besl solution was printed out). The total cost of the network is seen to have decreased from 493,776 cost units for the initial assumed flow distribution, to the final value of 479,525, a decrease of approximately 3%. Note that no initial design was assumed, but only the flow distribution, and the value of 493,776 is the optimal cost for that flow distribution. Had one been required to specify a first design, it is quite certain that its cost would have been considerably higher. In the final solution the head is exactly equal to the minimum required at three of the nodes, and higher at the others.

The optimal designs for the initial flow distribution and for the final flow distribution are shown in Figure 3. Note the tendency toward a branching design, which was constrained by the minimum diameter (I in.) and maximum allowable hydraulic gradient (0.05).

This simple network was studied extensively, to see how cost = LP(Q) changes with Q. The flows in the network were changed systematically by incrementing AQ_X and *&Qz,* the flow changes in the two loops from some initial flow distribution. The cost = LP(Q) was obtained for each new Q and was plotted versus *&Q,* and AQ_2. The response surface showed multiple local optimums, with low ridges along directions in the (A£>i, *SQz)* plane which correspond to zero flows in links, an indication of the low cost of branching configurations.

EXTENSION OF THE METHOD FOR COMPLEX SYSTEMS

Several types *of* variables did not appear in the formulation presented above. We now introduce them one by one and show how the basic formulation of the LPG method is made applicable to more general hydraulic systems.

Multiple loadings. In the design of a water distribution system, one should consider its operation under more than one loading. The maximum hourly flows and fire fighting demands are normally used as the design conditions, but often the low-demand periods, such as night flows, have to be considered as well. This is especially true when there is storage in the system. If only peak loads are considered in the design process, the reservoirs may be sized properly, and they may empty at acceptable rates during peak demands, but there is no guarantee that it will be possible to fill them during periods of low demand. The LPG method allows for simultaneous, consideration of several loadings, thereby ensuring proper design and operation of the system. For each of the loadings, one has to specify an initial flow distribution which satisfies continuity at all the nodes. Then for each loading the constraints on heads at nodes (equation (4)) and the path constraints (equation (7)) are formulated. Constraints (7) for open paths, i.e.. between fixed head nodes, which have to be formulated by proceeding from the high to the low head, may be written in opposite senses for the high- and low-demand loadings. The constraints for all loadings, together with the length constraints (equation (!)), are satisfied simultaneously in the linear program which is

TABLE *5b*. Final Results for the Design of the Network in Figure 2: Node Data for Iteration 18

Node	Friction Losses	Minimum Pressure Allowed	Existing Pressure	Dual Activity
		Pressure equation		
2	6.6	30.0	53.4	0.0
3	19.2	30.0	30.8	0.0
4	10.9	30.0	44.1	0.0
5	30.0	30.0	30.0	-638.
6	15.0	30.0	30.0	-0.977 E04
7	20.0	30.0	30.0	-0.321 E04
		Loop equation		
2	0.0			184.
7	-0.0			-85.1

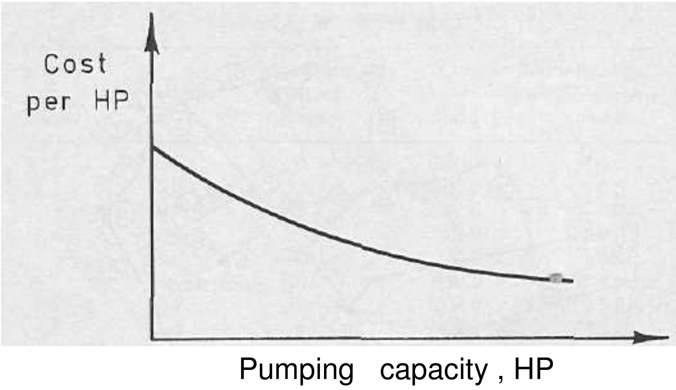


Fig. 4. Cos! per installed horsepower (schematic).

solved. The gradient section modifies the flow distribution for each of the loadings, using the results of the linear program.

Since the initial flows for each loading are arbitrary, one cannot guarantee that a set of diameters can be found such that the head line equations for all paths are balanced for all flow distributions. To overcome this difficulty, we introduce into each of the constraints of type (7) two new variables for each loading. These variables act essentially as valves; each provides a head loss when the flow in the path is in one of the two possible directions. These dummy valve variables make it possible to satisfy the head line constraints by 'operating' them differently for each load. These variables are given a large penalty in the objective function (and are therefore analogous to the artificial variables used in a linear program), and the optimization algorithm will try to eliminate them from the solution. If it is possible to do so, i.e., if there exists a feasible solution without these valves, then their introduction has merely served the purpose of reaching this feasible solution by the LP procedure. If. on the other hand, il is found that one o^ these dummy valves does appear in the optimal solution, this means that a real valve is needed at that point if the network is to operate as specified. The LP procedure deals simultaneously with reaching a hydraulically feasible design and optimizing it.

Pumps. When there are to be pumps in the system, the problem is one of the design-operation type; i.e., one has to select the capacities of the pumps as well as to decide which pumps should operate for each of the loading conditions. The locations at which pumps may be installed are selected by the designer, but since the program can set certain pump capacities to zero, if that is the optimal solution, the program actually selects the locations at which pumps will be installed. The decision variables associated with each location at which the designer has specified that a pump may be located are the heads to be added by the pump for each of the loadings. The maximum of these determines the pump capacity which has to be installed. If one denotes by *XP(t, I)* the head added by pump number *I* and load /, then the head constraints of the type (4) for paths with pumps become

$$H\ \min_n(l) \le H_s(l) \pm \sum_t XP(t,l) \pm \sum_{i,j} \sum_m J_{ijm}(l)x_{ijm} \le H\ \max_n(l) \qquad (17)$$

where the first summation is over the pumps in the path. An index / has been added to those variables which may be a function of the loading condition. For any path which has pumps in it. be it a closed loop or an open path, equation (7) has to be modified in a similar manner, and it then becomes

$$\sum_{i,j} \sum_m J_{ijm}(l)x_{ijm} \pm \sum_t XP(t,l) = b_p(l) \qquad (18)$$

In (17) and (18) the signs in front of the various terms depend on the direction of flow.

The decision variables for the pumps *XP(t, I)* have to be introduced linearly into the objective function if the problem is to remain a linear program. This is done by considering the cost of the pump as a function *o?* its capacity, i.e., its rated horsepower. Figure 4 shows schematically the cost per installed horsepower as a function of pump capacity. The curve we used in this work is based on real data and was found to have the shape seen in Figure 4. It reflects the decreasing marginal cost as the capacity increases. The actual cost data are introduced into the program, and there is no need to assume any particular form of this curve. Successive approximations are used in the program to cope with the nonlinearity of this cost curve. The power needed to operate the pump is given by

$$\text{hp} = \gamma \cdot Q \cdot XP/\eta \qquad (19)$$

where 7 is a coefficient, *Q* is the flow, *XP* is the head added by the pump, and *i)* is the efficiency. If we assume a fixed efficiency (we have used r\ = 0.75), then for a fixed discharge through the pump, (19) becomes

$$\text{hp} = K_t \cdot XP \qquad (20)$$

where *K,* is a constant. In equation (19), *y* is computed to

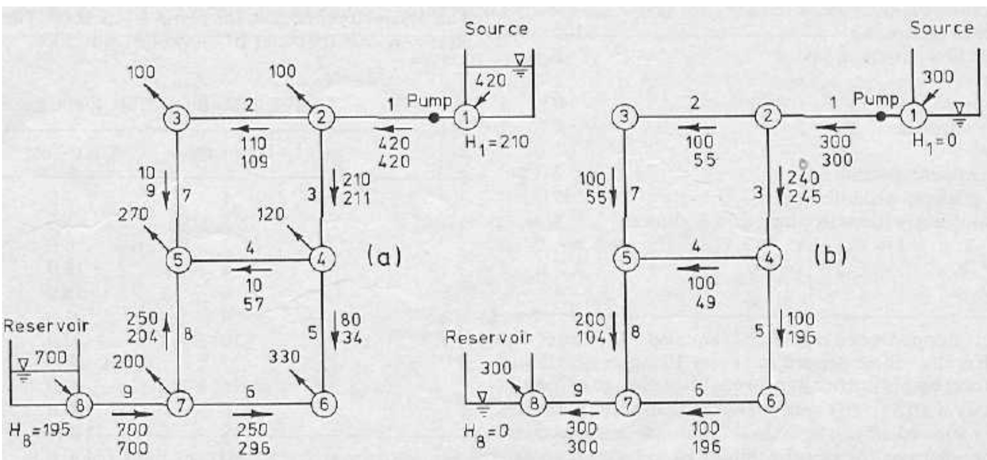


Fig. 5. Network with a pump, a balancing reservoir, loads, and initial and **final** flow distributions.

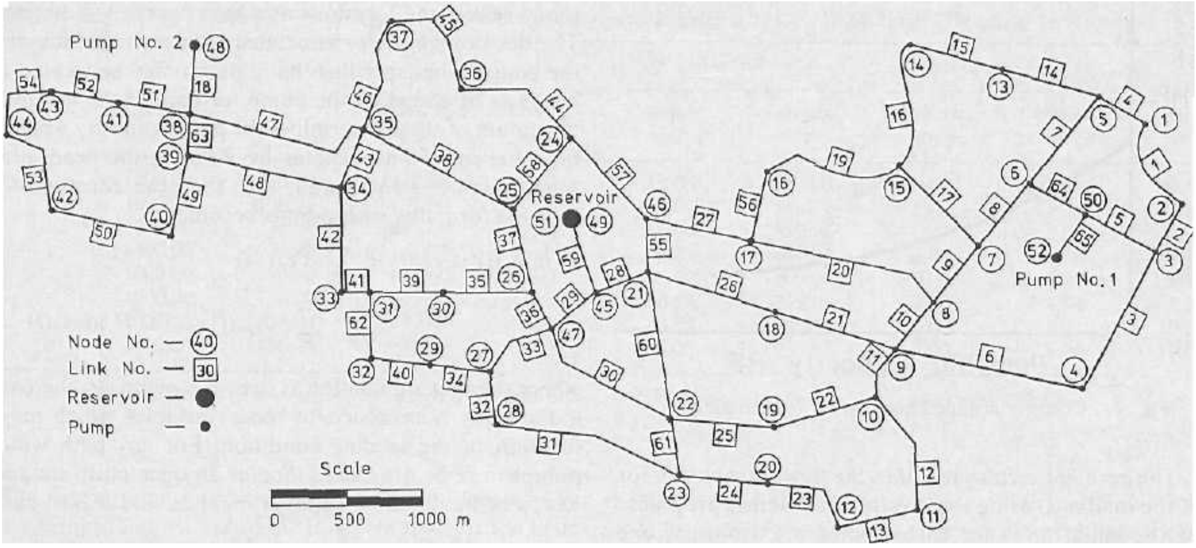


Fig. 6. Schematic diagram of a real network.

reflect the total length of time that the specific loading condition is assumed to prevail throughout the design horizon (say, 25 yr) and a coefficient that converts streams of annual expenditures to present value.

Recall now that the LPG procedure is to fix the discharges throughout the network, optimize, then change the flows. From (20), if the efficiency is assumed to be constant, the power is linearly proportional to the head added by the pump, for each linear program. The operating cost of the pump is therefore linearly proportional to the decision variable, which is *XP*. The capital cost, however, is not linear, as seen from Figure 4. This is where an iterative procedure is developed. (I) Assume values for the cost per horsepower, from the data represented in Figure 4, for each pump. (2) Solve the linear program with these values as the coefficients of the *XP* in the objective function. (3) For the resulting *XP* after the linear program has been solved, compute the cost per horsepower. I f all values are close enough to those assumed, this step is complete, and one proceeds to a flow iteration by the gradient method. Otherwise one takes the new costs and solves the linear program again.

This procedure has been found to work very well, owing probably to the relatively mild and regular slope of the cost curve. No more than 2-5 repetitions of the linear program at any gradient move were required to converge to within reasonable accuracy, with up to three pumps in the system. An alternative would have been to use separable programing, but due to the success of the relatively simple procedure outlined above, this was deemed unnecessary.

Each pump designed by this procedure may represent a pumping station. One now takes the values of the flow and

TABLE *6a.* Basic Design Data for The Network in Figure 5 With Two Loops, One Pump, One Source, One Reservoir, and Two Loads

	Value
Number of sections	9
Number of nodes	8
Greatest diameter allowed, in.	20
Maximum capacity of pumps, hp	100
Number of different flow distributions	2
Number of pumps	1
Number of valves	0
Number of dummy variables	6
Number of storages	1
Lowest allowable gradient, promilles	1.0
Highest allowable gradient, promilles	30.0
Number of loop/source equations in which flow is allowed to change	4
Initial step size, m³/h	5.0
Number of free loops	4

The computation stopped when max *DQ{!)* equaled 1.0 or after 50 major iterations. Results will be printed out every 10 major itcralions or whenever best total cost is further improved. Flow change in loop / will be executed only if *DQ(l)/DQ* max is greater than 0.15. Number of minor iterations allowed after a feasible solution has been reached for a flow distribution was 20, Local solution is considered to be reached if last improvement per iteration equaled 0.0%.

TABLE *6b.* Basic Pump Data: Cost Function of Pumps

hp	Cost
1	3000
11	3000
21	1800
31	1300
41	1000
51	850
100	500

Other data include the following. Pump 1 was connected to pipe 1. The assumed initial cost for pump 1 was 1000. The additional storage elevation cost {per unit of elevation) was 2000.

TABLE *6c.* Basic Cost Data for Pipes

Diameter	Unit Cost
1	2.0
2	5.0
3	8.0
4	11.0
6	16.0
8	24.0
10	32.0
12	50.0
14	60.0
16	90.0
18	130.0
20	170.0

TABLE *la.* Node Data

Node	Elevation, m	Minimum Pressure Allowed	Consumption, m*/h	
			Load 1	Load 2
1	210.0	0.0	-420.0	-300.0
2	150.0	30.0	100.0	0.0
3	160.0	30.0	100.0	0.0
4	155.0	30.0	120.0	0.0
5	150.0	30.0	270.0	0.0
6	165.0	30.0	330.0	0.0
7	160.0	30.0	200.0	0.0
8	195.5	0.0	-700.0	300.0

head for each loading and selects the pumps for this station, which will deliver these flows at the prescribed heads.

Valves and dummy valves. Valves may be located in any pipe. If one denotes by *XV(v, I)* the head loss provided by the valve at location *v* under the /th load, the appropriate constraints will contain this variable in the same way that *XP{t, I)* was in equations (14) and (15). The cost of the vaive should then be incorporated into the cost of the pipeline in which it is located.

When more than one loading is considered, two dummy valves have to be added in each loop, as was explained in a previous section. The variables *XV(i\ /)* of the dummy valve appear in the constraints in the same way as they would for real valves. These *XV* are given a high penalty in the objective function, which will tend to delete them from the optimal solution whenever this is possible.

Reservoirs. Systems having operational storage have lo be designed for more than one loading, since by definition the storage has to act as a buffer for the sources, i.e., to fill al times of low demand and then empty when demands peak. It should be mentioned in passing that proper design of the storage, i.e., its sizing, the way it is linked to the distribution system, and the way in which it is operated, is one of the most difficult tasks in design, one for which good engineering tools are missing. The storage is usually sized in accordance with some accepted standard, but it often does not perform its intended operational role, i.e., it stays at a relatively constant level, not really helping to balance the load on the sources. We think that the method presented here goes a long way toward solving this problem. The solution obtained is such that the reservoirs are not only sized but actually operated in an optimal manner.

The decision variable for a reservoir is the elevation at which it is to be located, An initial elevation is assumed, then *XR* is the additional elevation where the reservoir is to be located, relative to its initially assumed elevation. Path equations have lo be formed between the reservoir at node *s* and nodes in the network. For node *n,*

$$H\ \min_n(l) \le (HO_s + XR_s) \pm \sum_{i,j} \sum_m J_{ijm}(l) x_{ijm} \le H\ \max_n(l) \qquad (21)$$

TABLE *7b.* Section Data

Section	Length, m	C	Range of Allowable Diameters, in.	Class	Initial Flow Distribution, m³/h		Selected Diameters, in.
					Load 1	Load 2	
1	1000.0	130.0	0–20	1	420.0	300.0	10, 12, 14, 16, 18
2	1000.0	130.0	0–20	1	110.0	100.0	6, 8, 10
3	1000.0	130.0	0–20	1	210.0	200.0	8, 10, 12, 14
4	1000.0	130.0	0–20	1	10.0	100.0	6, 8, 10
5	1000.0	130.0	0–20	1	80.0	100.0	6, 8, 10
6	1000.0	130.0	0–20	1	250.0	100.0	8, 10, 12, 14, 16
7	1000.0	130.0	0–20	1	10.0	100.0	6, 8, 10
8	1000.0	130.0	0–20	1	250.0	200.0	8, 10, 12, 14, 16
9	100.0	130.0	0–20	1	700.0	300.0	12, 14, 16, 18, 20

TABLE *1c,* Structure of the Initial Linear Program: Strings of Pipes for Pressure or Loop Constraints

Begin Node	End Node	Load	Number Order of Sections Connected Between the Nodes	Number Order of Pumps, Valves, and Storages
			Pressure equation	
1	3	1	1, 2	−1
1	4	1	1, 3	−1
1	5	1	1, 3, 4	−1
1	6	1	1, 3, 5	−1
8	7	1	9	1
			Source equation	
1	8	1	1, 3, 5, −6, −9	−1, −2, 3, −1
1	8	2	1, 3, 5, 6, 9	−1, −2, 3, −1
			Loop equation	
2	2	1	3, 4, −7, −2	−4, 5
2	2	2	3, 4, −7, −2	−4, 5
4	4	2	5, 6, −8, −4	−6, 7
4	4	1	5, −6, 8, −4	−6, 7

The constraint data include 51 variables, iive pressure equations, two source equations, four loop equations and a coefficient matrix of 20 rows and 71 columns.

TABLE 8«. Optimal Solution for the Network in Figure 5: Section Data and Optimal Diameters for Iteration 22

Section	Length, m	C	Load 1, m³/h	Load 2, m³/h	Loop 1 Diameter, in.	Loop 1 Length, m	Loop 2 Diameter, in.	Loop 2 Length, m	HF1, m	HF2, m
1	1000.00	130.00	420.00	300.00	10.00	33.89	12.00	966.08	8.99	4.82
2	1000.00	130.00	109.08	55.50	5.00	374.46	8.00	625.51	10.91	3.12
3	1000.00	130.00	210.92	244.50	10.00	1000.00	0.0	0.0	5.82	7.65
4	1000.00	130.00	57.26	48.47	6.00	816.14	8.00	183.85	5.39	3.96
5	1000.00	130.00	33.65	196.03	10.00	999.97	0.0	0.0	0.19	5.08
6	1000.00	130.00	296.34	196.03	14.00	999.99	0.0	0.0	2.12	0.99
7	1000.00	130.00	9.08	55.50	6.00	929.62	4.00	70.36	0.30	8.49
8	1000.00	130.00	203.65	103.97	10.00	825.67	8.00	174.28	7.32	2.11
9	100.00	130.00	700.00	300.00	16.00	100.00	0.0	0.0	0.54	0.11

The solution was reached after one minor iteration. The total network cost (there were no penalty costs) was 299,851; the total pipeline cost, 267,113; the additional storage elevation cost, 4330; and the total pumping cost, 28,408. The total lengths of pipes Tor class 1 were ihe following: for diameters of 4, 6, 8, 10, 12, 14, and 16 in., the lengths were, respectively, 70.4, 2120.2, 983.6, 2859.5, 966.1, 1000, and 100 m.

where HO^* is the initial elevation of the reservoir at node 5, and XR_h is the additional elevation to be selected by the program. Note that XR_S is a single variable, the same for all loadings, since the reservoir once fixed cannot be moved. The coefficient of XR_S in the objective function is the cost of raising the location of the reservoir by one unit (1 m). An upper bound on XR_H is added when the topography allows only a certain range of elevations for the reservoir.

The flow into or out of the reservoir is specified in a manner similar to that of all other demands. The designer sets the flow out of the reservoir during peak demands and into the reservoir at low demands, and the program will find the optimal network configuration, including the pumps if they are included, which will operate the storage as required.

Combined systems. One can allow all types of elements in the network: pumps, reservoirs, and valves. Each path constraint will include the appropriate elements as explained in the preceding sections, and the solution will give simultaneously the optimal values of the decision variables: pipe diameters, pump capacities, and pump and valve operation for each loading, and the reservoir elevation.

Additions to an existing system. When parts of the network already exist and only new parts are to be designed, the existing components are specified as being fixed, and the program solves for the rest. For each existing pipe, its diameter is specified as being the only one on its candidate list. When entire loops exist, two dummy valves have to be inserted in each of them even when only one loading is considered. This insertion takes place because the initially assumed flow distribution will in general not be feasible hydraulically. The high penalty incurred by the 'operation' of these variables will guide the gradient procedure in modifying the flow distribution in existing loops toward the hydraulically correct values.

When a pump exists, the $XP(t, i)$ values are constrained to be less than its actual head capacity. The objective function will then contain only the cost of operating the pump over the design horizon.

Operation of an existing system. One can solve for the operational variables, pump operation and valve settings when the entire system is already in existence. The objective function now includes only the cost of operating the pumps (and penalties on operating the dummy valves).

TABLF- 8/). Optimal Solution for the Network in Figure 5: Node Data for Iteration 22

Node	Friction	Minimum Allowed Pressure	Existing Pressure	Dual
Pressure equation				
3	19.9	30.0	30.1	0.0
4	14.8	30.0	40.2	0.0
5	20.2	30.0	39.8	0.0
6	15.0	30.0	30.0	−0.556 E 04
7	−1.6	30.0	37.1	0.0
Source equation				
8	14.5			0.650 E 04
8	14.5			−0.450 E 04
Loop equation				
2	0.0			480.
2	0.0			120.
4	−0.0			0.294 E 04
4	−0.0			101.

SECOND EXAMPLE

Figure 5 shows a network similar to the one in Figure 2, with a pump added at the source and a reservoir linked to node 7 by an additional pipe. Basic data appear in Tables 6a-6c. There are two loadings, and six dummy valves are needed: one for each loading for each of the three equations, one between the two reservoirs, and one for each loop.

The cost for the pump as a function of its horsepower is given as a piecewise linear function: at 11 hp the value is 3000, at 21 hp the value is 1800, etc. As an initial value the cost is assumed to be 1000, i.e., a horsepower of 41. The cost for raising the reservoir at node 8 is 2000 per unit rise (1 m).

Node and pipe data, as well as the setup of the initial linear program, are given in Tables *la-lc*. Minimum head constraints are given for nodes 3, 4, 5, 6, and 7 under loading number 1 only, since it is assumed that at low demands, heads will be adequate. Two path equations are specified between the source and the reservoir. For example, the one for loading number 1 is

$$-\sum \Delta + \sum \Delta H_{6,7,m}(1) + \sum \Delta H_{7,8,m}(1) + XP(1,1) - XV(2,1) + XV(3,1) = H_8 - H_1 \quad (22)$$

where $\&H_{Um}(l)$ - $J_{Um}(l)x_{Um}$ is the head loss in the mth segment of the pipe connecting nodes *i* and *j* under the /th loading; $XP/\backslash, I)$ is the head added by the pump (whose number is 1) under the /th loading; and $XV(v, t)$ is the head

TABLE 9a. Basic Design Data for a Real Network With **Two** Loads

	Value
Number of sections	65
Number of nodes	52
Greatest diameter allowed, in.	20
Maximum capacity of pumps, hp	500
Number of different flow distributions	2
Number of pumps	2
Number of valves	0
Number of dummy variables	30
Number of storages	1
Lowest allowable gradient, promilles	0.5
Highest allowable gradient, promilles	25.0
Number of loop/source equations in which flow is allowed to change	34
Initial step size, m³/h	5.0
Number of free loops	34

The compulation stopped when max *OQ{1)* equaled 1.0 or after 30 major iterations. Results will be primed out every live major iterations or whenever best total cost is further improved. Flow change in loop / will be executed only if *DQ(I)/DQ* max is greater than 0.20. The number of minor iterations allowed after a feasible solution has been reached for a flow distribution was 160. The local solution is considered to be reached if last improvement per iteration equals 0.0%.

loss due to dummy valve number *v* under the /th loading. Two dummy valves appear in this equation, each able Lo produce a head loss when flow is in one of the two possible directions. The optimal solution, which was obtained in 22 flow iterations, is given in Tables 8a and 86, The total cost is 299.851, out of which 267,113 is the pipeline cost, 4330 is for raising the reservoir by 2.2 m, and 28,408 is the pumping cost. The pump is operated only under loading number 2, and adds 6.3 m to the head, above the head al the source. The cost of the optimal network for the initial flow distributions is 323.666, of which 294,350 is the pipeline cost. 9700 is for raising the reservoir by 4.9 m, and 19,616 is the pumping cost. The pump is operated only for loading number 2, and adds 4.4 m of head above the source. The process has thus reduced the total cost by over 7%, relative to the optimal cost for the initial flow distribution.

SUMMARY AND CONCLUSIONS

The main features of the LPG method for optimal design of water distribution systems, as presented in this paper, are the following.

1. The method deals with multiple loadings simultaneously.
2. Operational decisions are included explicitly in the design process.
3. Decision variables are pipe diameters, pump capacities, valve locations, reservoir elevations, and pump and valve operations for each loading.
4. The method yields a design that is hydraulically feasible for each of the loadings.
5. The design obtained is closer to being optimal than the one from which the search is started; this holds true even when the optimization procedure is terminated prematurely.
6. The method is applicable to real, complex systems.

Some weak points of the method are the following.

1. The engineer has to select the layout of the network, the location of pumps and reservoirs, and the initial How distribution. Even though the method can end with 'zero elements' (eliminate certain elements), thereby allowing some measure of selection between alternate system configurations, these can only be configurations which are 'close' to the one specified by the engineer.
2. The objective function contains only capital and operating costs. It should reflect other aspects as well, such as performance (for example, instead of imposing rigid constraints on pressures at supply nodes, one could use residual pressures as an additional performance criterion and include them in the objective function) and reliability (a more basic delinilion of reliability should be developed and made part of the objective function, instead of setting arbitrary constraints on the minimal diameter allowed for certain pipes).
3. Only a local optimum is reached by the search procedure. Several starling points have to be tried if one is lo have some assurance of not having missed a better design.
4. The search procedure relies on several heuristics whose efficient use requires experience.
5. Flows into and out of reservoirs have to be fixed for each of the loadings. This ensures proper operation *of* the reservoirs but does not include their capacity as a decision variable.

Several aspects uf the overall approach, the LPG method, and its implementation in the computer program are under improvement and further development. Among them are the following aspects.

1. A 'screening model'[1] is being investigated. Its task will be to propose the basic system configuration, which will then be optimized by the LPG method.
2. Instead of specifying the initial flow distribution, the engineer will be able lo specify the initial design. A network solver will solve for the flows in this network, and the optimization procedure will take over from there.
3. Several aspects of the optimization procedure are being investigated. The heuristics governing the step size and the termination criterion for the search procedure are under investigation. The termination criterion for the linear program within each flow iteration is being examined. This is aimed at preventing excessive computations after feasibility is reached in the linear program, since there is no need to reach exact LP optimality except on the last flow iteration. Other search methods for flow modilication are being examined, among theni a method which does not use the gradient.

APPENDIX 1: SOME DETAILS OV rat IMPLEMENTATION

Selection of the candidate diameters. At the outset the list of candidate diameters for each link is based on a minimum and a maximum value of the hydraulic gradient supplied to the program. We have normally used 0.0005 and 0.025 (or 0.050) for these values. For the initial assumed flow distribution, these limiting gradients will yield a maximum and a minimum diameter admissible for each link. All pipes in this diameter range, from a list supplied lo the program as data, are then put on the initial candidate list for the link. There is good reason to

TABLE 9b. Basic Pump Data: Cost Function of Pumps

hp	Cost
1	2000
100	2000
500	2000

Other data include the following. Pump I was connected lo pipe 65 and pump 2 to pipe 18. The assumed initial cost for both pumps was 2000. The additional storage elevation cost (per unit of elevation) for storage I was 22,000.

TABLE 9c. Basic Cost Data for Pipes

Diameter	Unit Cost
2	20.0
3	33.0
4	48.0
6	84.0
8	123.0
10	166.0
12	213.0
14	163.0
16	315.0
18	1000.0
20	439.0

specify a relatively narrow range, since it decreases the number of decision variables in the initial LP problem (the number will change later). At the same time, the limits cannot be too narrow, or the linear program will have no feasible solution and the computer run will fail.

If the first selection of candidate diameters resulted in an optimal LP solution the list may have to be modified after the flows in the network have been changed by the gradient move. The modifications in the lists of candidate diameters are based on the following rules. (I) If in the optimal LP solution a link is made entirely of one diameter, then for the next linear program the list is made of three diameters, the existing one and both its neighbors. (2) When in the optimal LP solution a link is made of two diameters, the list for the next linear program is made of these two, plus one adjacent to that diameter which has the longer of the two lengths.

Both cases result in a list of only three diameters for each link. In going from one LP solution to the next the list for a particular link may remain unchanged, or a diameter may be dropped off one end of the list and a new one added at the other.

Updating the LP matrix and its inoer.se between successive flow iterations. Since the coefficients in the LP matrix all depend on the flows in the network, they will all change after each flow iteration, A well-known technique for updating the inverse matrix, due to changes in the matrix itself, was implemented and proved to save considerable computer time. The same thing holds true upon introduction of a new candidate diameter to replace an old one. The column that belongs to this new diameter now has new values in it. As occurred before, the inverse is updated in an efficient way, which eliminates the need to reinvert.

The first basic solution for each linear program. Since the changes from one flow iteration to the next arc not major, it is reasonable to assume that the new optimal basis will contain much of the previous one. The number of iterations in each linear program can be kept low by starting it with the old basis, i.e., the same variables are in the starting basis. There are three possible cases. (I) This basis is optimal, (2) This basts is feasible but not optimal. A new LP iteration is started, and the process is continued to optimality. (3) This basis is not feasible. This is detected by computing the value of each row and finding that one or more of the RHS's are negative (when all are nonnegative the basis is feasible, and we are in the second case above). One then finds that row having the largest (absolute) value, and one subtracts this row from all other infeasible ones. This makes all other rows feasible, since now their value is positive. In the only row that is infeasible, one introduces an artificial variable, which is given a very high penalty in the objective function and is thus 'forced out' in the next iteration. These updating procedures have been found to keep the number of LP iterations down.

Computing times. The program was written in Fortran and was run on an IBM 370/168. Computing times for the three examples given in this paper were as follows: two loops, single loading (Figure 2), 19 iterations, 4.05 s cpu; two loops, two loadings (Figure 5), 22 iterations, 7.39 s cpu; and real network. Appendix 2, 10 iterations, 540 s cpu. These times do not include compilation (which required 11 s of cpu time), since the runs were made from the compiled program.

APPENDIX 2: DESIGN OF A REAL SYSTEM

The method was applied to the system shown in Figure 6, which has 51 nodes, 65 pipes. 15 loops, two pumps which

TABLE 9*d*. Node Data

Node	Elevation, m	Minimum Pressure Allowed	Consumption, m³/h Load 1	Consumption, m³/h Load 2
1	370.2	30.0	25.0	10.0
2	350.0	30.0	37.0	16.0
3	335.7	30.0	80.0	35.0
4	340.0	30.0	115.0	48.0
5	357.0	30.0	27.0	11.0
6	345.4	30.0	85.0	37.0
7	318.8	30.0	29.0	8.0
8	317.4	30.0	55.0	27.0
9	315.1	30.0	32.0	13.0
10	309.1	30.0	43.0	18.0
11	306.0	30.0	34.0	14.0
12	309.0	30.0	70.0	29.0
13	360.0	30.0	29.0	12.0
14	360.0	30.0	24.0	10.0
15	348.2	30.0	56.0	23.0
16	360.0	30.0	39.0	16.0
17	342.5	30.0	67.0	28.0
18	331.0	30.0	45.0	19.0
19	311.1	30.0	53.0	22.0
20	306.3	30.0	31.0	13.0
21	343.7	30.0	33.0	14.0
22	315.7	30.0	88.0	24.0
23	308.8	30.0	0.0	12.0
24	366.7	30.0	16.0	7.0
25	341.9	30.0	45.0	18.0
26	326.8	30.0	7.0	3.0
27	313.4	30.0	36.0	15.0
28	305.7	30.0	58.0	24.0
29	311.9	30.0	23.0	16.0
30	321.8	30.0	30.0	13.0
31	327.2	30.0	31.0	13.0
32	320.0	30.0	17.0	7.0
33	340.0	30.0	16.0	6.0
34	350.0	30.0	26.0	15.0
35	355.6	30.0	65.0	23.0
36	361.0	30.0	16.0	6.0
37	370.0	30.0	6.0	3.0
38	358.1	30.0	30.0	12.0
39	353.2	30.0	29.0	12.0
40	345.0	30.0	26.0	11.0
41	359.3	30.0	24.0	10.0
42	348.2	30.0	35.0	15.0
43	362.2	30.0	23.0	10.0
44	358.4	30.0	29.0	12.0
45	370.6	30.0	3.0	3.0
46	360.0	30.0	13.0	5.0
47	315.8	30.0	17.0	7.0
48	406.0	0.0	611.0	387.0
49	401.0	0.0	439.0	322.0
50	406.0	0.0	0.0	0.0
51	399.0	0.0	439.0	322.0
52	406.0	0.0	667.0	654.0

TABLH *9c.* Section Data

Section	Length, m	C	Range of Allowable Diameters, in.	Class	Initial Flow Distribution, m³/h		Selected Diameters, in.
					Load 1	Load 2	
1	640.0	130.00	3–20	1	10.0	5.0	3, 4, 6
2	485.0	130.00	3–20	1	47.0	21.0	4, 6, 8, 10
3	895.0	130.00	3–20	1	150.0	198.0	8, 10, 12, 14, 16
4	335.0	130.00	3–20	1	15.0	5.0	3, 4, 6
5	860.0	130.00	3–20	1	277.0	254.0	8, 10, 12, 14, 16
6	1285.0	130.00	3–20	1	35.0	150.0	6, 8, 10, 12, 14
7	715.0	130.00	3–20	1	151.0	81.0	6, 8, 10, 12, 14
8	570.0	130.00	3–20	1	154.0	282.0	8, 10, 12, 14, 16
9	420.0	130.00	3–20	1	105.0	244.0	8, 10, 12, 14, 16
10	370.0	130.00	3–20	1	32.0	189.0	8, 10, 12, 14, 16
11	390.0	130.00	3–20	1	25.0	212.0	8, 10, 12, 14, 16
12	855.0	130.00	3–20	1	14.0	78.0	6, 8, 10, 12
13	610.0	130.00	3–20	1	20.0	64.0	4, 6, 8, 10
14	705.0	130.00	3–20	1	109.0	65.0	6, 8, 10, 12
15	640.0	130.00	3–20	1	80.0	53.0	6, 8, 10, 12
16	385.0	130.00	3–20	1	56.0	43.0	4, 6, 8, 10
17	765.0	130.00	3–20	1	20.0	30.0	3, 4, 6, 8
18	150.0	130.00	3–20	1	611.0	387.0	12, 14, 16, 18, 20
19	890.0	130.00	3–20	1	20.0	50.0	4, 6, 8, 10
20	1260.0	130.00	3–20	1	18.0	28.0	3, 4, 6, 8
21	805.0	130.00	3–20	1	10.0	114.0	6, 8, 10, 12
22	685.0	130.00	3–20	1	32.0	116.0	6, 8, 10, 12
23	675.0	130.00	3–20	1	90.0	35.0	6, 8, 10, 12
24	570.0	130.00	3–20	1	121.0	22.0	6, 8, 10, 12, 14
25	660.0	130.00	3–20	1	85.0	94.0	6, 8, 10, 12
26	890.0	130.00	3–20	1	35.0	95.0	6, 8, 10, 12
27	710.0	130.00	3–20	1	68.0	34.0	4, 6, 8, 10
28	400.0	130.00	3–20	1	278.0	165.0	8, 10, 12, 14, 16
29	400.0	130.00	3–20	1	158.0	160.0	6, 8, 10, 12, 14
30	980.0	130.00	3–20	1	150.0	10.0	6, 8, 10, 12, 14
31	1150.0	130.00	3–20	1	7.0	20.0	3, 4, 6
32	365.0	130.00	3–20	1	65.0	4.0	4, 6, 8, 10
33	470.0	130.00	3–20	1	4.0	67.0	4, 6, 8, 10
34	325.0	130.00	3–20	1	105.0	86.0	6, 8, 10, 12
35	530.0	130.00	3–20	1	5.0	50.0	4, 6, 8, 10
36	280.0	130.00	3–20	1	5.0	90.0	6, 8, 10, 12
37	600.0	130.00	3–20	1	7.0	43.0	4, 6, 8
38	1085.0	130.00	3–20	1	70.0	50.0	4, 6, 8, 10
39	440.0	130.00	3–20	1	35.0	63.0	4, 6, 8, 10
40	495.0	130.00	3–20	1	128.0	96.0	6, 8, 10, 12, 14
41	300.0	130.00	3–20	1	211.0	179.0	8, 10, 12, 14, 16
42	630.0	130.00	3–20	1	227.0	185.0	8, 10, 12, 14, 16
43	330.0	130.00	3–20	1	14.0	10.0	3, 4, 6
44	900.0	130.00	3–20	1	5.0	23.0	3, 4, 6
45	770.0	130.00	3–20	1	21.0	29.0	3, 4, 6, 8
46	790.0	130.00	3–20	1	27.0	32.0	3, 4, 6, 8
47	1175.0	130.00	3–20	1	166.0	115.0	6, 8, 10, 12, 14
48	970.0	130.00	3–20	1	249.0	190.0	8, 10, 12, 14, 16
49	525.0	130.00	3–20	1	57.0	21.0	4, 6, 8, 10
50	785.0	130.00	3–20	1	31.0	10.0	3, 4, 6, 8
51	430.0	130.00	3–20	1	80.0	37.0	6, 8, 10, 12
52	460.0	130.00	3–20	1	56.0	27.0	4, 6, 8, 10
53	885.0	130.00	3–20	1	4.0	5.0	3, 4, 6
54	495.0	130.00	3–20	1	33.0	17.0	3, 4, 6, 8
55	395.0	130.00	3–20	1	74.0	34.0	4, 6, 8, 10
56	505.0	130.00	3–20	1	19.0	34.0	3, 4, 6, 8
57	765.0	130.00	3–20	1	7.0	5.0	3, 4, 6
58	585.0	130.00	3–20	1	18.0	11.0	3, 4, 6
59	150.0	130.00	3–20	1	439.0	322.0	10, 12, 14, 16, 18
60	960.0	130.00	3–20	1	136.0	50.0	6, 8, 10, 12, 14
61	370.0	130.00	3–20	1	113.0	10.0	6, 8, 10, 12
62	410.0	130.00	3–20	1	145.0	103.0	6, 8, 10, 12, 14
63	275.0	130.00	3–20	1	335.0	223.0	8, 10, 12, 14, 16
64	70.0	130.00	3–20	1	390.0	400.0	10, 12, 14, 16, 18
65	20.0	130.00	3–20	1	667.0	654.0	12, 14, 16, 18, 20

supply from external sources, and a balancing reservoir. (The computer output shows 52 nodes, because the reservoir is given two numbers, one for each loading. It shows 34 loop equations: 15 for loops and two for equations between the sources and the reservoir, one equation for each loading.) This network was designed several years ago to serve one out of

TABLB 9/ Structure of the Initial Linear Program: Strings of Pipes for Pressure or Loop Constraints

Begin Node	End Node	Load	Number Order of Sections Connected Between the Nodes	Number Order of Pumps, Valves, and Storages
			Pressure equation	
49	45	1	59	1
49	46	1	28, 59, 55	1
52	1	1	64, 7, 4, 65	−1
52	12	1	64, 8, 9, 10, 11, 12, 13, 65	−1
52	14	1	64, 7, 14, 15, 65	−1
52	16	1	19, 17, 8, 64, 65	−1
52	18	1	64, 8, 9, 10, 21, 65	−1
48	24	1	18, 47, 46, 45, 44	−2
48	32	1	18, 63, 48, 42, 41, 62	−2
48	37	1	18, 47, 46	−2
48	42	1	18, 52, 53, 54, 51	−2
48	44	1	51, 52, 54, 18	−2
52	1	2	64, 7, 4, 65	−1
52	12	2	64, 8, 9, 10, 11, 12, 13, 65	−1
52	14	2	64, 7, 14, 15, 65	−1
52	16	2	19, 17, 8, 64, 65	−1
52	18	2	64, 8, 9, 10, 21, 65	−1
48	24	2	18, 47, 46, 45, 44	−2
48	32	2	18, 63, 48, 42, 41, 62	−2
48	37	2	18, 47, 46	−2
48	42	2	18, 52, 53, 54, 51	−2
48	44	2	51, 52, 54, 18	−2
			Source equation	
52	49	1	64, 8, 9, 20, −27, −55, −28, −59	−1, 0, −1
52	51	2	64, 8, 9, 20, 27, 55, 28, 59, 65	−1, 0, −1
52	48	1	29, −36, −37, −38, −47, −18, 64, 8, 9	−1, 2
			20, −27, −55, −28, 65	
52	48	2	−29, −36, −37, −38, −47, −18, 65, 8, 9	
			20, 27, 55, 28, 65	−1, 2
			Loop equation	
1	1	1	64, 7, 4, −1, −2, −5	3, −4
9	9	1	−64, −8, −9, −10, 5, 3, 6	7, −8
17	17	1	9, 20, −17, −19, 56	9, −10
17	17	1	−20, 10, 21, −26, 55, 27	11, −12
12	12	1	−23, −24, −61, 25, 22, 12, −13	13, −14
22	22	1	−30, −29, 28, 60	15, −16
28	28	1	30, 61, −31, −32, 33	17, −18
45	45	1	58, 57, −55, 29, −28, −36, −37	19, −20
19	19	1	−25, −60, 26, −21, 11, −22	21, −22
24	24	1	44, 45, 46, −38, −58	23, −24
34	34	1	−43, 38, 37, −35, −39, −42, −41	25, −26
35	35	1	47, −63, −48, 43	27, −28
42	42	1	−51, −52, −53, −54, 49, 50, 63	29, −30
31	31	1	39, 35, 36, −33, −34, −40, −62	31, −32
1	1	2	64, 7, 4, −1, −2, −5	3, −4
15	15	2	8, 17, −7, −14, −15, −16	5, −6
9	9	2	−64, −8, −9, −10, 5, 3, 6	7, −8
17	17	2	9, 20, −17, −19, −56	9, −10
17	17	2	−20, 10, 21, 26, −55, −27	11, −12
12	12	2	23, 24, −61, −25, −22, 12, 13	13, −14
22	22	2	−30, −29, 28, 60	15, −16
28	28	2	−30, 61, 31, −32, 33	17, −18
45	45	2	−58, 57, 55, −29, 28, −36, −37	19, −20
19	19	2	25, 60, −26, −21, 11, −22	21, −22
24	24	2	44, 45, 46, −38, −58	23, −24
34	34	2	−43, 38, 37, −35, −39, −42, −41	25, −26
42	42	2	−51, −52, −53, −54, 49, 50, 63	29, −30
31	31	2	39, 35, 36, −33, −34, −40, −62	31, −32
15	15	1	8, 17, −7, −14, −15, −16	5, −6
35	35	2	47, −63, −48, 43	27, −28

The constraint data include 340 variables, 22 pressure equations, four source equations, 30 loop equations, and a coefficient matrix of 121 rows and 461 columns.

TABLE 9*g*. Optimal Solution for the Real Network: Section Data and Optimal Diameters for Iteration 2

					Loop 1		Loop 2			
Section	Length, m	*C*	Load 1, m³/h	Load 2, m³/h	Diameter, in.	Length, m	Diameter, in.	Length, m	HF1, m	HF2, m
1	640.00	130.00	10.00	5.00	4.00	639.99	0.0	0.0	1.14	0.32
2	485.00	130.00	47.00	21.00	10.00	484.99	0.0	0.0	0.17	0.04
3	895.00	130.00	150.00	198.00	8.00	11.77	14.00	883.22	0.64	1.07
4	335.00	130.00	15.00	5.00	3.00	102.98	6.00	232.02	1.70	0.22
5	860.00	130.00	277.00	254.00	8.00	100.59	14.00	759.40	4.30	3.66
6	1285.00	130.00	35.00	150.00	8.00	978.35	14.00	306.64	0.62	9.16
7	715.00	130.00	149.75	82.24	14.00	714.99	0.0	0.0	0.43	0.14
8	570.00	130.00	155.25	280.76	16.00	569.99	0.0	0.0	0.19	0.57
9	420.00	130.00	101.60	244.00	8.00	420.00	0.0	0.0	1.87	9.49
10	370.00	130.00	32.00	189.00	16.00	370.00	0.0	0.0	0.01	0.18
11	390.00	130.00	26.55	212.00	8.00	170.63	6.00	219.36	0.39	18.47
12	855.00	130.00	15.14	78.00	6.00	854.99	0.0	0.0	0.46	9.48
13	610.00	130.00	18.86	64.00	10.00	610.00	0.0	0.0	0.04	0.39
14	705.00	130.00	107.75	66.24	12.00	705.00	0.0	0.0	0.49	0.20
15	640.00	130.00	78.75	54.24	12.00	640.00	0.0	0.0	0.25	0.12
16	385.00	130.00	54.75	44.24	10.00	385.00	0.0	0.0	0.18	0.12
17	765.00	130.00	24.65	28.76	6.00	76.64	8.00	688.35	0.32	0.43
18	150.00	130.00	611.00	387.00	14.00	150.00	0.0	0.0	1.22	0.52
19	890.00	130.00	23.40	50.00	4.00	466.78	6.00	423.20	4.52	18.43
20	1260.00	130.00	14.60	28.00	4.00	734.00	6.00	525.98	2.90	9.67
21	805.00	130.00	8.45	114.00	12.00	805.00	0.0	0.0	0.00	0.62
22	685.00	130.00	31.59	116.00	6.00	288.95	8.00	396.04	0.80	8.94
23	675.00	130.00	88.86	35.00	12.00	675.00	0.0	0.0	0.33	0.06
24	570.00	130.00	119.86	22.00	14.00	570.00	0.0	0.0	0.23	0.01
25	660.00	130.00	84.59	94.00	8.00	51.73	10.00	608.26	0.82	0.99
26	890.00	130.00	36.55	95.00	6.00	537.89	8.00	352.11	1.70	9.98
27	710.00	130.00	68.00	34.00	10.00	709.99	0.0	0.0	0.51	0.14
28	400.00	130.00	278.00	165.00	14.00	146.70	16.00	253.29	0.53	0.20
29	400.00	130.00	158.00	160.00	16.00	11.69	14.00	388.30	0.26	0.27
30	980.00	130.00	150.00	13.24	10.00	58.78	14.00	921.20	0.74	0.01
31	1150.00	130.00	7.00	16.76	6.00	1150.00	0.0	0.0	0.15	0.74
32	365.00	130.00	65.00	7.24	6.00	252.69	10.00	112.30	2.07	0.04
33	470.00	130.00	4.00	63.76	10.00	470.00	0.0	0.0	0.00	0.30
34	325.00	130.00	105.00	86.00	12.00	325.00	0.0	0.0	0.21	0.15
35	530.00	130.00	5.00	50.00	10.00	530.00	0.0	0.0	0.00	0.21
36	280.00	130.00	5.00	90.00	12.00	280.00	0.0	0.0	0.00	0.14
37	600.00	130.00	7.00	43.00	10.00	142.86	8.00	457.13	0.02	0.46
38	1085.00	130.00	66.38	55.00	12.00	768.02	8.00	316.97	0.86	0.61
39	440.00	130.00	35.00	63.00	8.00	60.14	10.00	379.85	0.12	0.35
40	495.00	130.00	128.00	96.00	14.00	495.00	0.0	0.0	0.22	0.13
41	300.00	130.00	211.00	179.00	16.00	300.00	0.0	0.0	0.18	0.13
42	630.00	130.00	227.00	185.00	16.00	630.00	0.0	0.0	0.43	0.29
43	330.00	130.00	14.00	10.00	6.00	330.00	0.0	0.0	0.15	0.08
44	900.00	130.00	8.62	18.00	6.00	900.00	0.0	0.0	0.17	0.66
45	770.00	130.00	24.62	24.00	8.00	770.00	0.0	0.0	0.25	0.24
46	790.00	130.00	30.62	27.00	4.00	32.46	6.00	757.53	1.95	1.54
47	1175.00	130.00	166.00	115.00	6.00	38.50	8.00	1136.50	14.31	7.25
48	970.00	130.00	249.00	190.00	8.00	109.40	14.00	860.60	3.89	2.36
49	525.00	130.00	57.00	21.00	6.00	525.00	0.0	0.0	3.26	0.51
50	785.00	130.00	31.00	10.00	4.00	784.99	0.0	0.0	11.36	1.40
51	430.00	130.00	80.00	37.00	6.00	303.86	4.00	126.14	14.10	3.38
52	460.00	130.00	56.00	27.00	6.00	460.00	0.0	0.0	2.76	0.72
53	885.00	130.00	4.00	5.00	4.00	885.00	0.0	0.0	0.29	0.44
54	495.00	130.00	33.00	17.00	4.00	495.00	0.0	0.0	8.04	2.35
55	395.00	130.00	74.00	34.00	4.00	1.68	6.00	393.32	4.08	0.97
56	505.00	130.00	15.60	34.00	8.00	504.99	0.0	0.0	0.07	0.30
57	765.00	130.00	7.00	5.00	3.00	764.99	0.0	0.0	2.86	1.53
58	585.00	130.00	14.38	16.00	4.00	406.56	6.00	178.44	1.51	1.83
59	150.00	130.00	439.00	322.00	14.00	150.00	0.0	0.0	0.66	0.37
60	960.00	130.00	134.45	50.00	14.00	960.00	0.0	0.0	0.47	0.08
61	370.00	130.00	111.86	6.76	6.00	41.26	10.00	328.74	1.48	0.01
62	410.00	130.00	145.00	103.00	14.00	410.00	0.0	0.0	0.23	0.12
63	275.00	130.00	335.00	223.00	14.00	214.35	6.00	60.65	10.57	4.98
64	70.00	130.00	390.00	400.00	10.00	8.01	8.00	61.99	3.48	3.65
65	20.00	130.00	667.00	654.00	14.00	20.00	0.0	0.0	0.19	0.18

The solution was reached after 43 improvement iterations; in all there were 45 minor iterations. The tolal network cost, including penalty costs was 1,403,999,488; the total network cost excluding penalties, 5.722,635: the total pipeline cost, 5,440,668: additional storage elevation cost, 51,233; and (he total pumping cost, 230.735.

four pressure zones within a city whose total population is forecast for the end of the design period at 200,000, Elevations in this pressure zone range from 1295 m at the lower right to 1370 m at the top left, with a hill rising to 1450 m where the reservoir is shown. Residual pressures are to be at least 30 m at all nodes. (Only 22 constraints on minimum pressures at selected nodes were specified: 12 for the peak loading, load 1, and 10 for the low-demand loading, load 2.)

The design was based on the projected peak hourly demands, which total 1720 mVh (1.1 mgd), and a low-demand period with a total demand of 725 m^a/h (0,46 mgd), during which the reservoir was to fill. The reservoir capacity was set at 6900 m^3 (1.68 mg) by using reserve and fire-fighting considerations.

Difficulties were encountered in the engineering design, even though a network solver was used extensively. The main difficulty was in utilization of the reservoir to balance the load on the sources. A satisfactory design was reached only after considerable trial-and-error work. Costs from the original design were not available, and therefore the optima! cost of the LPG design cannot be compared with that of the engineering design. It is clear, however, that a satisfactory hydraulic design has been reached, and that it is cheaper than the design on the first iteration.

Tables *9a-9g* show the design and cost data, the setup of the first linear program, and the optimal solution. The computer run cost approximately S60 and resulted in a cost reduction from 6,263,747 on the first flow iteration to 5,722,635 at the optimal solution, a reduction of approximately 9%.

Acknowledgment. C. D. Howard and P. E. Flatt, of Howard and Associates, Winnipeg, made many useful suggestions on the presentation of the method and the format of the paper. The computer program is available from the second author.

REFERENCES

Gupta, 1., Linear programming analysis of a water supply system, *Trans, Amer. Inst. Ind. Eng., /*(I), 56-61, 1969.

Gupta, L, M. Z. Hassan, and J. Cook, Linear programming analysis of a water supply system with multiple supply points, *Trans. Amer. Inst. Ind. Eng.,* 4(3), 200-204, 1972.

Hamberg, D., Optimal location of pumping stations In a branching network (in Hebrew). M.Sc. thesis, Fac. of Civil Eng., Technion-lsrael Inst, of Technol., Haifa, July 1974.

Jacoby, S. L. S., Design of optimal hydraulic network, *J. Hvdraul. Div. Amer. Soc. Civil Eng.,* W(HY3), 641-661, 1968.

Kally. E., Computerized planning of the least cost water distribution network, *Water Sewage Works,* 121-127, 1972.

Karmeli, D., Y. Gadish, and S. Meyers, Design of optimal water distribution networks, *J. Pipeline Div. Amer. Soc. Civil Eng.,* W(PLI), 1 — 10, 1968.

Kohlhaas, C, and D. E. Mattern, An algorithm for obtaining optimal looped pipe distribution networks, in *Papers of the 6th Annual Symposium on the Application of Computers to the Problems of the Urban Society,* pp. 138-151, Association ofComputing Machinery, New York, 1971.

Lai, D., and J. C. Schaake, Linear programming and dynamic programming application to water distribution network design, 3, Engi--. neering systems analysis of the primary water distribution network of New York City, 1 1 1 pp., Dep. of Civil Eng., Mass. Inst, of Technol., Cambridge, July 1969.

Rasmusen. H. J,, Simplified optimization of water supply systems, *Environ. Eng. Div. Amer. Soc. Civil Eng.,* /02(EE2), 313-327, 1976.

Shamir, U., Water distribution systems analysis, *Rep. RC 4389,* 290 pp., IB M Thomas J. Watson Res. Center, Yorktown Heights, N. Y., 1973,

Shamir, U., Optimal design and operation of water distribution systems. *Water Resour. Res., 11(4),* 27-36, 1974.

Shamir, U., and C, D. D. Howard, Water distribution systems analysis. *J. Hvdraul. Div. Amer. Soc. Ciuil Eng.,* 94{HY1), 219-234, 1968,

Watanatada, T., Least cost design of water distribution systems, *J. Hvdraul. Div. Amer. Soc. Civil Eng., 99{HY9),* 1497-1513. 1973.

(Received December 28, 1976;
revised April 26, 1977;
accepted April 26, 1977.)

Comment, on 'Design of Optimal Water Distribution Systems' by E. Alperovits and U. Shamir

G. E. QUINDRY, E. D. BRILL, JR., J. C. LIEBMAN, AND A. R. ROBINSON

Department of Civil Engineering, University of Illinois at Urbana-Champaign, Vrbana, Illinois 61801

Alperovits and Shamir [1977] have provided a valuable contribution to the problem of optimal distribution system design. They have suggested an iterative approach which uses the dual variables from the solution of a suboptimal linear program to determine gradients with which the linear program is modified. Since we believe the approach is a significant advancement, we are attempting to integrate it into our own research activities. Alperovits and Shamir have provided us with a preliminary version of their computer program. In working with this material we find that there are mathematical corrections in the gradient derivation which are required to calculate the gradient and to apply the gradient properly to subsequent iterations.

In the derivation of the gradient for path p as given in (9)-(12) of the original paper the interactions of the paths with each other have been neglected. As a result, required terms are omitted from (12). A derivation of the gradient which includes these extra terms is presented below. An additional, and less important, point is that the sign of each gradient term, as presented in the original paper, appears to be incorrect. This can be seen by examining Table 4 of the original paper, if $G_1 = \partial(\text{cost})/\partial(\Delta Q_1)$, as defined in (12) of the paper, and G_1 is positive, a positive ΔQ_1, should result in a positive $\Delta(\text{cost})$; in such cases, however, $\Delta(\text{cost})$ is actually negative. This difficulty may be more a matter of presentation than of substance, since the computer program provided by Alperovits and Shamir correctly applies their gradient, once the additional terms are included.

The additional terms required in the gradient calculations are necessary because a change in flow in one path affects the flow in other paths in the network. For example, consider the network presented in Figure 2 in the original paper. It is a two-loop network, and link 4 is in each loop. Each loop is a path as defined in (7), and a change in flow in one loop affects the flow in the other loop. In addition, there are six paths defined by the six minimum node pressure constraints, and these must also be considered when these constraints are binding and can therefore have nonzero dual variables. Although these paths must be considered in calculating gradients for other paths, it is not necessary to calculate a gradient for these paths specifying minimum node pressure.

The gradients should be calculated using the following replacement for (12):

$$G_p = \frac{\partial(\text{cost})}{\partial(\Delta Q_p)} = -W_p \sum_{i,j\in p} (1/Q_{ij}) \sum_m \Delta H_{ijm} + \sum_{r\in R} \pm W_r \sum_{\substack{i,j\in r\\ i,j\in p}} (1/Q_{ij}) \sum_m \Delta H_{ijm} \qquad (1)$$

where R is used to denote all the paths other than p in the network and the sign of each additional term depends on whether or not path r uses link i, j in the same direction as path p. The sign is negative if the direction is the same and positive otherwise.

As an example, the information in Table 1 of this comment can be used to calculate the correct gradient for iteration 18 of the example problem. That iteration is described in Figure *3b* and Tables 4, *5a,* and *5b* of the original paper. Loop A includes links 3 and 4 in the direction of flow and links 2 and 7 in the opposite direction. With the exception of the path to node 2, every path in Table 1 includes at least one of the links in loop A and must therefore be included in the calculation of the gradient, $\partial(\text{cost})/\partial(\Delta Q_A)$. The equation for the gradient component would be

$$\frac{\partial(\text{cost})}{\partial(\Delta Q_A)} = -184\left(\frac{4.32}{843.00} + \frac{19.11}{121.39} + \frac{10.83}{77.00} + \frac{12.60}{177.00}\right) + (-85.1)\left(\frac{19.11}{121.39}\right) + 0\left(\frac{12.60}{177.00}\right) - 0\left(\frac{4.32}{843.00}\right) - (-638)\left(\frac{4.32}{843.00} + \frac{19.11}{121.39}\right) - (-9770)\left(\frac{4.32}{843.00}\right) - (-3210)\left(\frac{4.32}{843.00}\right) \qquad (2)$$

or

$$\frac{\partial(\text{cost})}{\partial(\Delta Q_A)} = +87.94$$

Similarly, the gradient component for loop B can be obtained as follows:

$$\frac{\partial(\text{cost})}{\partial(\Delta Q_B)} = +85.1\left(\frac{4.11}{601.60} + \frac{5.00}{271.60} + \frac{10.00}{71.61} + \frac{19.11}{121.39}\right) - (-184)\left(\frac{19.11}{121.39}\right) + (-638)\left(\frac{19.11}{121.39}\right) - (-9970)\left(\frac{4.11}{601.60}\right) - (-3210)\left(\frac{4.11}{601.60} + \frac{5.00}{271.60}\right) \qquad (3)$$

or

$$\frac{\partial(\text{cost})}{\partial(\Delta Q_B)} = +103.73$$

Since each gradient is positive, both ΔQ_A and ΔQ_B should be negative.

Alperovits and Shamir's computer program was modified to incorporate the new gradient calculation for the small example problem. The programing change is specific to that

Paper number 9W 1051
0043-1397/79/009W-1051$01.00

TABLE 1. Link and Path Data for Interation 18 of Example Problem

Section	Q_{ij}	$\sum_m \Delta H_{ijm}$
1	1120.0	6.57
2	177.00	12.60
3	843.00	4.32
4	121.39	19.11
5	601.60	4.11
6	271.60	5.00
7	77.00	10.83
8	71.61	10.00
Path	W_p	**Links Included**
Loop A	184.	3, 4, −7, −2
Loop B	−85.1	5, 6, 8, −4
to node 2	0.	1
to node 3	0.	1, 2
to node 4	0.	1, 3
to node 5	−638.	1, 3, 4
to node 6	−9770.	1, 3, 5
to node 7	−3210.	1, 3, 5, 6

problem and does not yield a general purpose program. Because of the complexity of the program no attempt at a general purpose correction was made. Using the starting solution given by Alperovits and Shamir (iteration 1 in their Table 4) and the same initial step size, the intermediate results shown in Table 2 were obtained. The final cost of 441,522 (using the same cost units specified in the original paper) represents a considerable improvement from the 479,525 obtained and presented in the original paper.

Derivation of Gradient Expression

For illustrative purposes the gradient expression given in general form in (1) is derived for the example problem shown in Figure 1. For simplicity, only the two loop paths are considered. It is assumed that additional paths, which would be used to specify minimum node pressures, are not included and therefore such paths are not considered in the derivation of the gradient expressions. In an actual problem, of course these paths must be taken into account, as shown in (2) and (3). New notation is defined as follows:

- Hl_{ij} head loss in the link between nodes i and j in the direction of the flow;
- Q_{ij} flow in the link between i and j;
- $d\alpha$ change in flow in all the links in loop A;
- $d\beta$ change in flow in all the links in loop B;
- γ head discontinuity at node 4 from loop A;
- δ head discontinuity at node 5 from loop B.

The signs of the flows, head losses, changes in flow, and changes in head losses are all positive in the direction of the arrows in Figure 1. Solving the linear program results in the following: (1) the set of X_{ijm} values which give the minimum cost for the given flow pattern, (2) the cost of that solution and (3) the set of dual variables, which are associated with pipe length, minimum heads at the nodes, and pressure along the paths in the network,

$$\frac{\partial(\text{cost})}{\partial L_{ij}} \qquad \frac{\partial(\text{cost})}{\partial H\text{MIN}_h} \qquad \frac{\partial(\text{cost})}{\partial b_p}$$

TABLE 2. Example Problem With Corrected Gradient

Flow Iterations	Number of Linear Programing Iterations	Total Cost of Network	Loop A G_A	Loop A ΔQ_A	Loop B G_B	Loop B ΔQ_B
1	32	493,776	+60.9	−2.23	+137.	−5.00
2	1	492,190	+55.2	−1.79	+154.	−5.00
3	3	490,663	+50.6	−1.70	+149.	−5.00
4	6	489,684	+155.	−5.00	+54.2	−1.75
5	3	488,028	+62.3	−1.94	+161.	−5.00
6	5	486,853	+168.	−5.00	+66.1	−1.96
7	3	484,994	+73.4	−2.10	+175.	−5.00
8	5	483,605	+180.	−5.00	+82.3	−2.28
9	3	481,474	+83.6	−2.15	+195.	−5.00
10	5	479,888	+189.	−5.00	+107.	−2.83
11	1	477,473	+218.	−5.00	+95.5	−2.19
12	8	475,511	+93.5	−2.15	+218.	−5.00
13	5	473,283	+193.	−5.00	+121.	−3.15
14	5	471,577	+75.1	−2.82	+133.	−5.00
15	4	470,226	+193.	−5.00	+20.7	−0.536*
16	7	469,181	+87.	−3.32	+131.	−5.00
17	1	467,389	+83.6	−2.92	+143.	−5.00
18	3	465,611	+212.	−5.00	+22.2	−0.524*
19	9	464,751	+96.3	−3.39	+142.	−5.00
20	1	462,769	+92.	−2.89	+159.	−5.00
21	1	460,724	+87.2	−2.43	+180.	−5.00
22	7	459,213	+142.	−5.00	+82.4	−2.90
23	3	457,424	+35.6	−0.827*	+215.	−5.00
24	4	456,201	+122.	−3.63	+168.	−5.00
25	1	453,492	+117.	−2.33	+250.	−5.00
26	3	451,057	+103.	−3.41	+151.	−5.00
27	1	448,749	+98.5	−2.25	+219.	−5.00
28	1	445,655	+89.1	−1.12	+398.	−5.00
29	3	443,031	+77.4	−1.36	+284.	−5.00
30	2	443,079	+75.8	−0.151*	−1510.	3.00
31	11	441,522				

*Using the same rules as Alperovits and Shamir, these changes are not made because $\Delta Q_p < 0.2\Delta Q_{\max}$.

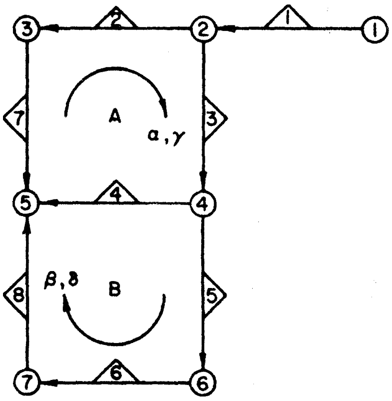


Fig. 1. Example network.

The loop constraints from the linear program can be written in the direction shown in Figure 1,

$$Hl_{45} - Hl_{35} - Hl_{23} + Hl_{24} = 0 \quad (4)$$

$$-Hl_{45} + Hl_{46} + Hl_{67} + Hl_{75} = 0 \quad (5)$$

The dual variables for these constraints, W_A and W_B, respectively, represent the change in cost that would result if the head loss in the loops could change. It is not possible for this change to occur in the actual problem, but it is useful to consider, in a mathematical sense, the imposition of a change in the head loss constraints. If a discontinuity in head loss is introduced, W_A and W_B can be used to determine whether the discontinuities in the right-hand sides of the constraints should be positive or negative to reduce the cost of the system. If the appropriate changes in head loss are made and the linear program resolved, the resulting network would not be hydraulically balanced. In the physical system, changes in the flows in the links are necessary to balance the head losses (i.e., to 'heal' the discontinuity). Since flow and head loss are not independent, the necessary changes in flow can be predicted. We wish to reverse this process, to change the flow (which is initialized before formulating the linear program) in such a manner that cost is reduced. It is important to note that the change in flow is not introduced to impose a change in head loss but is to correct a head loss discontinuity.

What is needed is an indicator of the direction and magnitude that cost will change with a change in the flow pattern, i.e., $\partial(\text{cost})/\partial\alpha$ and $\partial(\text{cost})/\partial\beta$. If these values are known, it is possible to calculate the changes in flow in each loop for any suitable step size. Once α and β are calculated, the change in flow in each link can be calculated. If the change in flow in each link is noted as dQ_{ij},

$$dQ_{23} = -d\alpha \quad (6)$$

$$dQ_{24} = d\alpha \quad (7)$$

$$dQ_{45} = d\alpha - d\beta \quad (8)$$

$$dQ_{46} = d\beta \quad (9)$$

$$dQ_{67} = d\beta \quad (10)$$

$$dQ_{35} = -d\alpha \quad (11)$$

$$dQ_{75} = d\beta \quad (12)$$

The flow pattern is completely determined by the known nodal inflows and outflows and the loop flows (α and β in the sample of Figure 1). What is desired is the gradient of the cost in the space of the loop flows, since the loop flows are the only variables that can be changed to go from one linear programing problem to the linear programing problem of the next iteration.

First, the discontinuity expressions are written explicitly:

$$Hl_{45} - Hl_{35} - Hl_{23} + Hl_{24} = \gamma \quad (13)$$

$$-Hl_{45} + Hl_{46} + Hl_{67} + Hl_{75} = \delta \quad (14)$$

We wish to determine the flow changes necessary to cancel γ and δ; the total change in head loss around the loops due to the changes in flows should be $-\gamma$ and $-\delta$, respectively. Thus the total increment in the head loss around each loop should be zero.

$$\frac{\partial Hl_{45}}{\partial Q_{45}} dQ_{45} - \frac{\partial Hl_{35}}{\partial Q_{35}} dQ_{35} - \frac{\partial Hl_{23}}{\partial Q_{23}} dQ_{23} + \frac{\partial Hl_{24}}{\partial Q_{24}} dQ_{24} + d\gamma = 0 \quad (15)$$

$$-\frac{\partial Hl_{45}}{\partial Q_{45}} dQ_{45} + \frac{\partial Hl_{46}}{\partial Q_{46}} dQ_{46} + \frac{\partial Hl_{67}}{\partial Q_{67}} dQ_{67} + \frac{\partial Hl_{75}}{\partial Q_{75}} dQ_{75} + d\delta = 0 \quad (16)$$

Substituting for each dQ_{ij} from (6)–(12),

$$\frac{\partial Hl_{45}}{\partial Q_{45}} (d\alpha - d\beta) - \frac{\partial Hl_{35}}{\partial Q_{35}} (-d\alpha) - \frac{\partial Hl_{23}}{\partial Q_{23}} (-d\alpha) + \frac{\partial Hl_{24}}{\partial Q_{24}} d\alpha + d\gamma = 0 \quad (17)$$

$$-\frac{\partial Hl_{45}}{\partial Q_{45}} (d\alpha - d\beta) + \frac{\partial Hl_{46}}{\partial Q_{46}} d\beta + \frac{\partial Hl_{67}}{\partial Q_{67}} d\beta + \frac{\partial Hl_{75}}{\partial Q_{75}} d\beta + d\delta = 0 \quad (18)$$

All the information required to determine the change in cost with respect to a change in the flow in the paths is now available,

$$\frac{\partial(\text{cost})}{\partial\alpha} = \frac{\partial(\text{cost})}{\partial\gamma} \cdot \frac{\partial\gamma}{\partial\alpha} + \frac{\partial(\text{cost})}{\partial\delta} \cdot \frac{\partial\delta}{\partial\alpha} \quad (19)$$

$$\frac{\partial(\text{cost})}{\partial\beta} = \frac{\partial(\text{cost})}{\partial\gamma} \cdot \frac{\partial\gamma}{\partial\beta} + \frac{\partial(\text{cost})}{\partial\delta} \cdot \frac{\partial\delta}{\partial\beta} \quad (20)$$

The terms $\partial(\text{cost})/\partial\gamma$ and $\partial(\text{cost})/\partial\delta$ are the dual variables for the loops, and the remaining terms can be found by picking out the coefficients of the differentials in (17) and (18),

$$\frac{\partial\gamma}{\partial\alpha} = -\frac{\partial Hl_{45}}{\partial Q_{45}} - \frac{\partial Hl_{35}}{\partial Q_{35}} - \frac{\partial Hl_{23}}{\partial Q_{23}} - \frac{\partial Hl_{24}}{\partial Q_{24}} \quad (21)$$

$$\frac{\partial\gamma}{\partial\beta} = +\frac{\partial Hl_{45}}{\partial Q_{45}} \quad (22)$$

$$\frac{\partial\delta}{\partial\alpha} = +\frac{\partial Hl_{45}}{\partial Q_{45}} \quad (23)$$

$$\frac{\partial\delta}{\partial\beta} = -\frac{\partial Hl_{45}}{\partial Q_{45}} - \frac{\partial Hl_{46}}{\partial Q_{46}} - \frac{\partial Hl_{67}}{\partial Q_{67}} - \frac{\partial Hl_{75}}{\partial Q_{75}} \quad (24)$$

Substituting these parital derivatives and the dual variables into (19) and (20), $\partial(\text{cost})/\partial\alpha$ and $\partial(\text{cost})/\partial\beta$ can be expressed as

$$\frac{\partial(\text{cost})}{\partial\alpha} = -W_A\left(\frac{\partial Hl_{45}}{\partial Q_{45}} + \frac{\partial Hl_{35}}{\partial Q_{35}} + \frac{\partial Hl_{23}}{\partial Q_{23}} + \frac{\partial Hl_{24}}{\partial Q_{24}}\right) + W_B\left(-\frac{\partial Hl_{45}}{\partial Q_{45}}\right) \quad (25)$$

$$\frac{\partial(\text{cost})}{\partial\beta} = W_A\left(\frac{\partial Hl_{45}}{\partial Q_{45}}\right) - W_B\left(\frac{\partial Hl_{45}}{\partial Q_{45}} + \frac{\partial Hl_{46}}{\partial Q_{46}} + \frac{\partial Hl_{67}}{\partial Q_{67}} + \frac{\partial Hl_{75}}{\partial Q_{75}}\right) \quad (26)$$

From the Hazen-Williams formula,

$$Hl_{ij} = \hat{\alpha}\left(\frac{Q_{ij}}{C}\right)^{1.852}(D_m)^{-4.87}L_{ij} \quad (27)$$

where D_m is the pipe diameter, L_{ij} is the length, and $\hat{\alpha}$ and C are constants. Taking the derivative with respect to Q_{ij},

$$\frac{\partial Hl_{ij}}{\partial Q_{ij}} = 1.852\hat{\alpha}(Q_{ij})^{0.852}C^{-1.842}(D_m)^{-4.87}L_{ij} = 1.852\,\frac{Hl_{ij}}{Q_{ij}} \quad (28)$$

The constant appears in each term of the gradient equations and can be omitted. It is therefore possible to write the gradients using only the dual variables and the ratio of head loss to flow in each link,

$$G_A = \frac{\partial(\text{cost})}{\partial\alpha} = -W_A\left(\frac{Hl_{45}}{Q_{45}} + \frac{Hl_{35}}{Q_{35}} + \frac{Hl_{23}}{Q_{23}} + \frac{Hl_{24}}{Q_{24}}\right) + W_B\left(\frac{Hl_{45}}{Q_{45}}\right) \quad (29)$$

$$G_B = \frac{\partial(\text{cost})}{\partial\beta} = W_A\left(\frac{Hl_{45}}{Q_{45}}\right) - W_B\left(\frac{Hl_{45}}{Q_{45}} + \frac{Hl_{46}}{Q_{46}} + \frac{Hl_{67}}{Q_{67}} + \frac{Hl_{75}}{Q_{75}}\right) \quad (30)$$

A generalization of the above derivation leads to the gradient equation given previously as (1).

A minor additional comment applies to the authors' statement that at the optimum of each linear programing problem, 'each link will contain at most two segments, their diameters being adjacent on the candidate list.' A counter example can be obtained by changing the cost of 16-inch diameter pipe in Table 1*b* of the original paper from 90.0 to 125.0 and solving the small example problem. The modification of the cost input results in the use of 14- and 18-inch pipe in one of the links, and these diameters are not adjacent. This point does not affect the overall method of optimization, but would require, perhaps only in such specially contrived cases, the modification of the heuristic employed for selecting feasible pipe diameters. Limiting the number of feasible diameters to three, as suggested, would seem to be too restrictive for links with non-adjacent diameters.

In conclusion, we believe the solution method provided by Alperovits and Shamir offers a promising advance in the field of optimal distribution system design. What is required is a further exploration of the interrelationships between the constraints of the linear programming formulation for both looped and branched systems.

REFERENCE

Alperovits, E., and U. Shamir, Design of optimal water distribution systems, *Water Resour. Res.,* 13(6), 885-900, 1977.

(Received April 10, 1978;
revised October 25, 1978;
accepted November 15, 1978.)

Reply

E. ALPEROVITS AND U. SHAMIR

Environment and Water Resources Engineering, Technion-Israel Institute of Technology, Haifa, Israel

We are grateful to *Quindry et al.* [1979] for their comments. Their correction of the procedure for computing the gradient indeed improves the performance of the linear programming gradient method for the examples tested. Quindry et al. provided us with the comments during their work with our programs, and while they implemented the correction only for the specific example cited, we have modified the general program to perform the gradient calculations as revised *[Quindry et al.,* 1979, equation (1)]. The new version of the program also incorporates numerous other modifications and improvements, which resulted from use in the design of a number of real water distribution systems.

As Quindry et al. state, the matter of the signs of the gradient terms is one of notation and presentation rather than one of substance. To generalize, the following statement can be made: Denote by F the objective function. Then $\partial(F)/\partial(\Delta Q_p) > 0$ indicates that $\Delta F > 0$ for $\Delta Q_P > 0$. If F is to be minimized (as is the case for cost, the objective function considered in the original paper), the gradient term for changing Q_p must be given the opposite sign of the derivative, i.e., $G_p = -\partial(F)/\partial(\Delta Q_p)$; while if F is to be maximized (for example, if F is some measure of system performance), then $G_p = d(F)/\partial(\Delta Q_p)$.

REFERENCE

Quindry, G. E., E. D. Brill., Jr., J. C. Liebman, and A. R. Robinson, Comment on 'Design of optimal water distribution systems' by E. Alperovits and U. Shamir, *Water Resour. Res., 15,* this issue, 1979.

(Received May 23, 1979; accepted July 2, 1979.)

Paper number 9W 1052.
0043-1397/79/009W-1052$01.00