Forecasting Water Levels in Aquifers by Numerical and Semihybrid Methods

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Two methods which employ a cell model for forecasting water levels in aquifers are compared: the (noniterative) alternating direction implicit (ADI) finite difference method and a semihybrid iterative method, in which a resistor network is the analog part. By using simulation of the semihybrid method it is concluded that this method requires a larger computational effort than the ADI method.

NEED FOR AQUIFER MODELS

Various questions related to the future groundwater regime have to be answered by the planning engineer, or hydrologist, in the process of determining the policy for operating an aquifer. Among these questions we may list those related to the quantity, location, and time of pumping from the aquifer and/or artificially recharging it with imported water, the need for additional pumping and recharge installations, the quality deterioration (e.g. by seawater intrusion or encroachment from adjacent aquifers), etc. Obviously, it is impossible to carry out experiments and tests in the aquifer itself in order to forecast its response to operations proposed in the future and to make comparisons among responses to different possible policies in order to determine the most desirable one. As in all other branches of science and engineering, whenever the treatment of real systems or phenomena is impossible (or the cost of such treatment is prohibitive), models of the systems or phenomena are used.

Here the term model is introduced in its most general sense. Often the terms conceptual model or mathematical model are used. It is a simplification of the complicated reality. There is no need to elaborate on the fact that most real systems, and certainly the aquifer system considered here, are indeed complicated beyond our capability to treat them as they really are. The porous medium continuum is inhomogeneous, anisotropic, etc., and simplifications are necessary. These take the form of a set of assumptions, which should not be forgotten whenever the model is being employed in the course of investigations. Examples of assumptions are that the flow is essentially horizontal, that water is released from storage in a phreatic aquifer immediately upon decline of the water table, or that the water table is a surface which separates a fully saturated region from a region with no moisture at all.

On the basis of such assumptions, the model of the investigated groundwater system is presented in the form of a set of mathematical equations, the solution of which yields the behavior of the considered system.

The choice of a model for a given aquifer system is dictated not only by the features of the aquifer itself (e.g., its geological properties) but also by the following criteria. First, it should be sufficiently simple so as to be amenable to mathematical treatment. Second, it should not be so simple that features of interest in the investigation on hand are excluded from it. As the range of possible models between these two limits is still wide, we should add two more important criteria, namely, that information should be available for calibrating the model and the model should be the most economic one for solving the problem on hand.

There is no justification for choosing a very detailed model, which should give very accurate results, if it cannot be properly calibrated so as to give the user confidence that the model indeed simulates the behavior of the real system. It is meaningless to seek a model which gives very accurate results when the input data are much less accurate, sometimes by far. Similarly, it is useless to choose a model which yields very detailed results when these can never be checked in the future against the behavior of the real aquifer system.

FINITE DIFFERENCE MODEL

The finite difference model is based on the following partial differential equation, which describes two-dimensional essentially horizontal flow in an anisotropic nonhomogeneous aquifer:

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial \phi}{\partial y}\right) + r - p = S\frac{\partial \phi}{\partial t} \qquad (1)$$

where T is the transmissivity; ϕ is the piezometric head, or potential; r is the rate of recharge (natural and/or artificial) per unit (horizontal) area; p is the rate of pumping per unit (horizontal) area; and S is the storativity.

Development of (1), and the assumptions on which it is based, can be found in most references on groundwater hydrology [e.g., *Bear,* 1972]. The equation is applicable to confined aquifers and to phreatic aquifers in which the drawdown is small in comparison to the thickness of the saturated zone below the phreatic surface. In this case the transmissivity at a point is approximated as the product of the average height (h) of the saturated zone and the average hydraulic conductivity (K) at the point.

To complete the mathematical statement of the flow problem in an aquifer, we have to state (1) the geometry of the aquifer's boundaries, e.g., $F = F(x, y)$, (2) the initial conditions for ϕ in the aquifer, e.g., $\phi = \phi(x, y, 0) = g(x, y)$, and (3) the boundary conditions which ϕ has to satisfy on F, e.g., ϕ = const, $\phi = f_1(x, y, t)$, $\partial\phi/\partial n = f_2(x, y, t)$, etc. Once the problem is thus stated and the values of $p = p(x, y, t)$, $r = r(x, y, t)$, $T = T(x, y)$, and $S = S(x, y)$ are specified, we seek $\phi = \phi(x, y, t)$, i.e., the future water levels resulting from p and r.

Equation (1) is solved numerically by the noniterative alternating direction implicit method, given by the following set of two equations:

$$\frac{1}{\Delta x_i}\left(T_{i+1/2,j}\frac{\phi_{i+1,j}{}^{n+1/2} - \phi_{i,j}{}^{n+1/2}}{\Delta x_{i+1/2}}\right.$$

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$$- T_{i-1/2,i} \frac{\phi_{i,i}^{n+1/2} - \phi_{i-1,i}^{n+1/2}}{\Delta x_{i-1/2}}\Bigg)$$

$$+ \frac{1}{\Delta y_i}\left(T_{i,i+1/2} \frac{\phi_{i,i+1}^{n} - \phi_{i,i}^{n}}{\Delta y_{i+1/2}}\right.$$

$$\left. - T_{i,i-1/2} \frac{\phi_{i,i}^{n} - \phi_{i,i-1}^{n}}{\Delta y_{i-1/2}}\right)$$

$$= S_{i,i} \frac{\phi_{i,i}^{n+1/2} - \phi_{i,i}^{n}}{0.5\ \Delta t^{n+1/2}} + p_{i,i}^{n+1/2} - r_{i,i}^{n+1/2} \quad (2)$$

$$\frac{1}{\Delta x_i}\left(T_{i+1/2,i} \frac{\phi_{i+1,i}^{n+1/2} - \phi_{i,i}^{n+1/2}}{\Delta x_{i+1/2}}\right.$$

$$\left. - T_{i-1/2,i} \frac{\phi_{i,i}^{n+1/2} - \phi_{i-1,i}^{n+1/2}}{\Delta x_{i-1/2}}\right)$$

$$+ \frac{1}{\Delta y_i}\left(T_{i,i+1/2} \frac{\phi_{i,i+1}^{n+1} - \phi_{i,i}^{n+1}}{\Delta y_{i+1/2}}\right.$$

$$\left. - T_{i,i-1/2} \frac{\phi_{i,i}^{n+1} - \phi_{i,i-1}^{n+1}}{\Delta y_{i-1/2}}\right)$$

$$= S_{i,i} \frac{\phi_{i,i}^{n+1} - \phi_{i,i}^{n+1/2}}{0.5\Delta t^{n+1/2}} + p_{i,i}^{n+1/2} - r_{i,i}^{n+1/2} \quad (3)$$

The computer programs which were developed for solving these equations are included in a report by *Hefez et al.* [1972]. To examine computation time and memory requirements for the alternating direction implicit (ADI) method, we have performed tests on models of sizes varying between 100 and 2500 cells. Runs were made on two computers. Results are shown in Figure 1 (it is interesting to note the tenfold reduction in computation time in going from the IBM 360/50 to the 370/165).

These results of the ADI method were used for comparison with the results of a semihybrid method, to be presented next.

ITERATIVE SOLUTION WITH A SEMIHYBRID COMPUTER

The hybrid solution is essentially a method for solving the implicit equation:

$$\frac{1}{\Delta x_i}\left(T_{i+1/2,i} \frac{\phi_{i+1,i}^{n+1} - \phi_{i,i}^{n+1}}{\Delta x_{i+1/2}}\right.$$

$$\left. - T_{i-1/2,i} \frac{\phi_{i,i}^{n+1} - \phi_{i-1,i}^{n+1}}{\Delta x_{i-1/2}}\right)$$

$$+ \frac{1}{\Delta y_i}\left(T_{i,i+1/2} \frac{\phi_{i,i+1}^{n+1} - \phi_{i,i}^{n+1}}{\Delta y_{i+1/2}}\right.$$

$$\left. - T_{i,i-1/2} \frac{\phi_{i,i}^{n+1} - \phi_{i,i-1}^{n+1}}{\Delta y_{i-1/2}}\right)$$

$$= S_{i,i} \frac{\phi_{i,i}^{n+1} - \phi_{i,i}^{n}}{\Delta t^{n+1/2}} + p_{i,i}^{n+1/2} - r_{i,i}^{n+1/2} \quad (4)$$

The hybrid computer combines an electric network of resistors with a digital computer and solves the discrete time water level forecasting problem. The solution is obtained by iterations in each time step. The hybrid approach was developed by *Karplus* [1964, 1965, 1966], with the stability conditions given by *Karplus and Kanus* [1965]. *Vemuri and Dracup* [1967] used the hybrid approach for forecasting water levels in a phreatic aquifer. They, as well as *Brandt* [1969], claim that the hybrid computer can compete with a pure numerical solution on a digital computer. To investigate this claim, we have used a simulated hybrid computer and have compared its performance with that of a numerical solution by the ADI method. We describe next the basic principles of the hybrid computation and then present results and draw conclusions on the comparison between the two methods.

The hybrid method. The hybrid computer is used to solve (4) for the water levels ϕ^{n+1} at the end of the time interval. First, multiply (4) by $\Delta x_i \Delta y_j$ and rewrite it in the form

$$\frac{\phi_{i-1,i}^{n+1} - \phi_{i,i}^{n+1}}{W_{i-1/2,i}} + \frac{\phi_{i+1,i}^{n+1} - \phi_{i,i}^{n+1}}{W_{i+1/2,i}}$$

$$+ \frac{\phi_{i,i-1}^{n+1} - \phi_{i,i}^{n+1}}{W_{i,i-1/2}} + \frac{\phi_{i,i+1}^{n+1} - \phi_{i,i}^{n+1}}{W_{i,i+1/2}}$$

$$= S_{i,i}\cdot\Delta x_i\cdot\Delta y_i \frac{\phi_{i,i}^{n+1} - \phi_{i,i}^{n}}{\Delta t^{n+1/2}}$$

$$+ QP_{i,i}^{n+1/2} - QR_{i,i}^{n+1/2} \quad (5)$$

where

$$QP_{i,j} = p_{i,j}\cdot\Delta x_i\cdot\Delta y_j$$

$$QR_{i,j} = r_{i,j}\cdot\Delta x_i\cdot\Delta y_j$$

and W is the 'resistance' between adjacent cells, e.g.,

$$W_{i-1/2,j} = \Delta x_{i-1/2}/\Delta y_j\cdot T_{i-1/2,j}$$

Equation (5) is solved by an iterative procedure. For convergence one must add to each side of (5) the quantity $-\phi_{i,j}^{n+1}/w_{i,j}$ [*Karplus and Kanus,* 1965], resulting in

$$\frac{\phi_{i-1,i}^{n+1} - \phi_{i,i}^{n+1}}{W_{i-1/2,i}} + \frac{\phi_{i+1,i}^{n+1} - \phi_{i,i}^{n+1}}{W_{i+1/2,i}}$$

$$+ \frac{\phi_{i,i-1}^{n+1} - \phi_{i,i}^{n+1}}{W_{i,i-1/2}} + \frac{\phi_{i,i+1}^{n+1} - \phi_{i,i}^{n+1}}{W_{i,i+1/2}} - \frac{\phi_{i,i}^{n+1}}{W_{i,i}}$$

$$= S_{i,i}\cdot\Delta x_i\cdot\Delta y_i \frac{\phi_{i,i}^{n+1} - \phi_{i,i}^{n}}{\Delta t^{n+1/2}} - \frac{\phi_{i,i}^{n+1}}{W_{i,i}}$$

$$+ QP_{i,i}^{n+1/2} - QR_{i,i}^{n+1/2} \quad (6)$$

The iteration procedure is based on (6), written in the following form, in which k is the iteration counter for computing ϕ^{n+1}:

$$\frac{\phi_{i-1,i}^{k+1} - \phi_{i,i}^{k+1}}{W_{i-1/2,i}} + \frac{\phi_{i+1,i}^{k+1} - \phi_{i,i}^{k+1}}{W_{i+1/2,i}}$$

$$+ \frac{\phi_{i,i-1}^{k+1} - \phi_{i,i}^{k+1}}{W_{i,i-1/2}} + \frac{\phi_{i,i+1}^{k+1} - \phi_{i,i}^{k+1}}{W_{i,i+1/2}} - \frac{\phi_{i,i}^{k+1}}{W_{i,i}}$$

$$= \left(S_{i,i} \frac{\Delta x_i\cdot\Delta y_i}{\Delta t^{n+1/2}} - \frac{1}{W_{i,i}}\right)\phi_{i,i}^{k} + QP_{i,i}^{n+1/2}$$

$$- QR_{i,i}^{n+1/2} - S_{i,i} \frac{\Delta x_i\cdot\Delta y_i}{\Delta t^{n+1/2}}\phi_{i,i}^{n} \quad (7)$$

The equation has been arranged so that the five unknowns appear on the left and the entire right-hand side is known. The iterations proceed as follows:

1. Assume the values of $\phi_{i,j}$ at the end of the time interval, and introduce them as $\phi_{i,j}^k$ in the right-hand side.
2. Compute $\phi_{i,j}^{k+1}$ in all cells by using the electric analogy to be explained below.
3. Introduce the new values in the right-hand side of (7) as $\phi_{i,j}^k$, and compute new values of $\phi_{i,j}^{k+1}$.

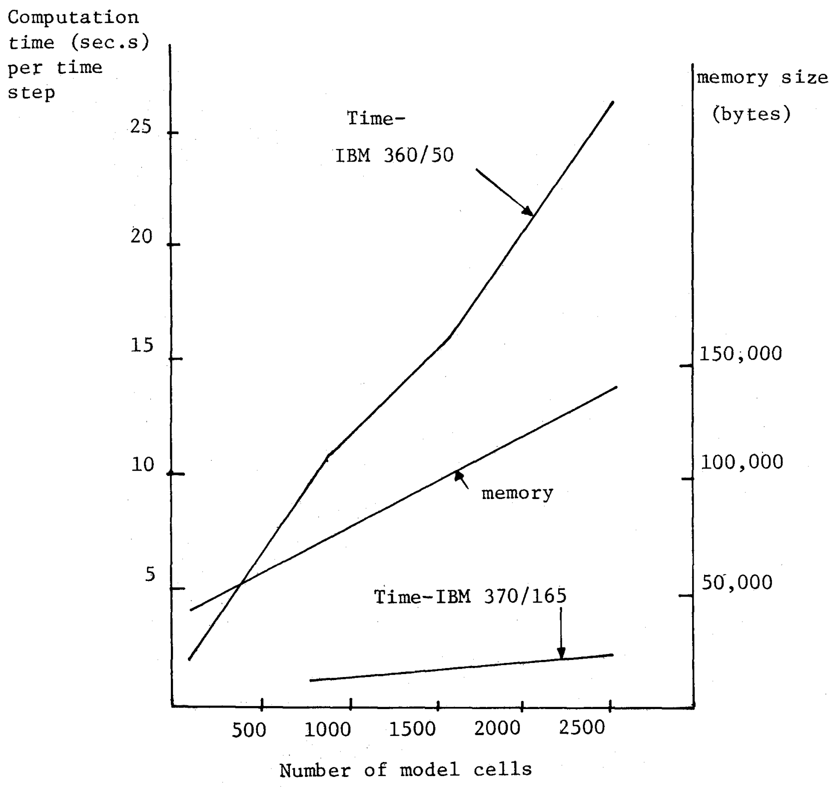

Fig. 1. Memory requirements and computation time per time step as a function of model size.

4. Repeat (2) and (3) until the values obtained in successive iterations differ by no more than some preset tolerance.

The value $W_{i,j}$ which was introduced above is chosen such that the following condition holds:

$$W_{i,j} \leq 2\Delta t^{n+1/2}/S_{i,j}\cdot\Delta x_i\cdot\Delta y_j \qquad (8)$$

The resistor network. If the known right-hand side of (7) is denoted as $-\Phi_{i,j}{}^k/W_{i,j}$, the equation can be rewritten as follows

$$\frac{\phi_{i-1,j}{}^{k+1}-\phi_{i,j}{}^{k+1}}{W_{i-1/2,j}}+\frac{\phi_{i+1,j}{}^{k+1}-\phi_{i,j}{}^{k+1}}{W_{i+1/2,j}}+\frac{\phi_{i,j-1}{}^{k+1}-\phi_{i,j}{}^{k+1}}{W_{i,j-1/2}}+\frac{\phi_{i,j+1}{}^{k+1}-\phi_{i,j}{}^{k+1}}{W_{i,j+1/2}}+\frac{\Phi_{i,j}{}^{k}-\phi_{i,j}{}^{k+1}}{W_{i,j}}=0 \qquad (9)$$

Equation (9) is solved by an electric analog made of resistors. Each grid resistor represents the resistance between adjacent cells of the model. An additional resistor is connected to each node, the other end being maintained at a voltage $U_{i,j}$. A typical node is shown in Figure 2.

The voltages at the nodes $V_{i,j}$ are the analogs of water levels in the aquifer $\phi_{i,j}$. Kirchhoff's law for a typical node is

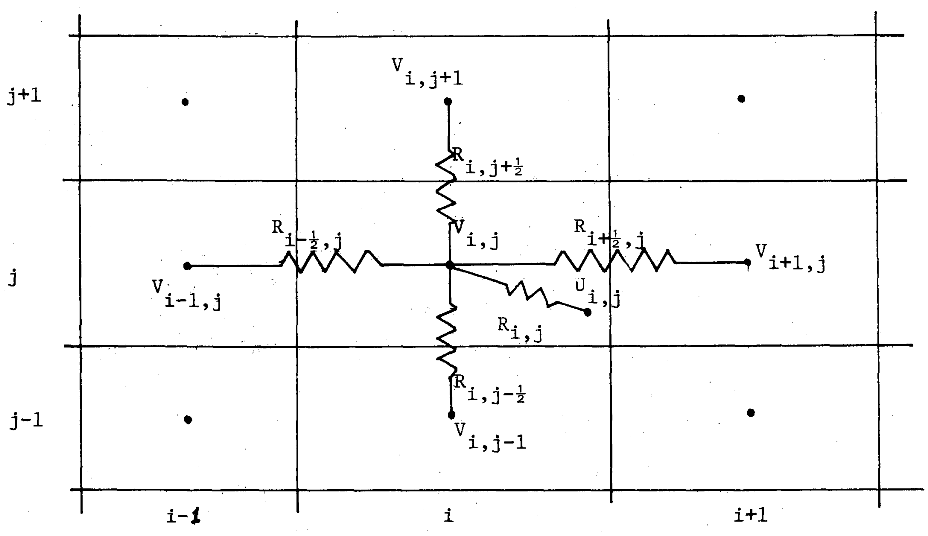

Fig. 2. A typical node of the electric resistor network.

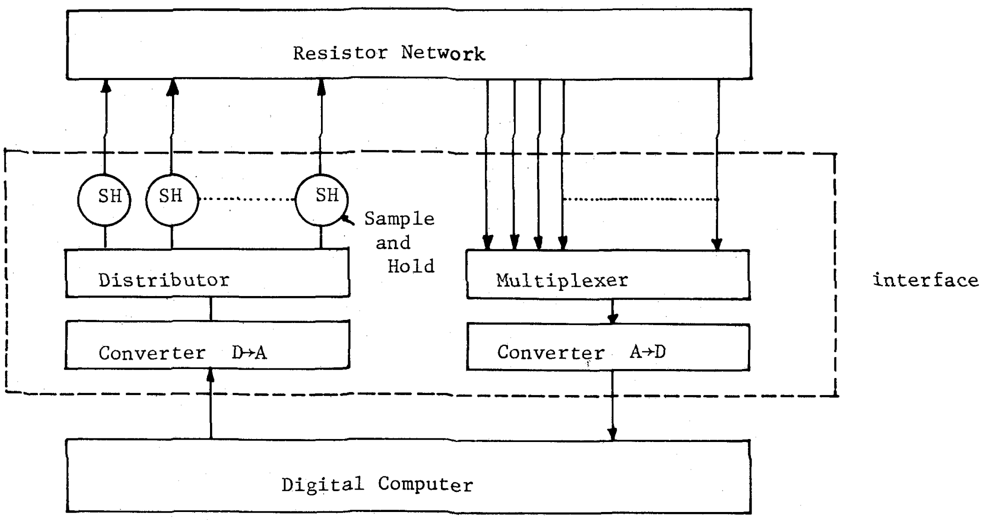

Fig. 3. Components of the hybrid computer.

$$\frac{V_{i-1,j}^{k+1} - V_{i,j}^{k+1}}{R_{i-1/2,j}} + \frac{V_{i+1,j}^{k+1} - V_{i,j}^{k+1}}{R_{i+1/2,j}} + \frac{V_{i,j-1}^{k+1} - V_{i,j}^{k+1}}{R_{i,j-1/2}} + \frac{V_{i,j+1}^{k+1} - V_{i,j}^{k+1}}{R_{i,j+1/2}} + \frac{U_{i,j}^{k} - V_{i,j}^{k+1}}{R_{i,j}} = 0 \qquad (10)$$

The analogy between the water levels and the voltages is obvious when (9) and (10) are compared. Thus if $U_{i,j}^k$ is set properly, the values of $\phi_{i,j}^{k+1}$ can be obtained by measuring the values of $V_{i,j}^{k+1}$ in the resistor network.

It can be shown [*Hefez,* 1972, Appendix A] that the resistance ratio

$$R_r = \frac{R_{i,j}}{W_{i,j}} = \frac{R_{i-1/2,j}}{W_{i-1/2,j}} = \frac{R_{i+1/2,j}}{W_{i+1/2,j}} = \frac{R_{i,j-1/2}}{W_{i,j-1/2}} = \frac{R_{i,j+1/2}}{W_{i,j+1/2}} \qquad (11)$$

can be chosen arbitrarily. Also, if one selects some typical difference in water levels $\Delta\phi$ and a corresponding voltage difference ΔV, then the ratio

$$V_r = \Delta V/\Delta\phi = U_{i,j}/\Phi_{i,j} \qquad (12)$$

can also be selected arbitrarily. There is, however, a restriction on the input resistors $R_{i,j}$, which stems from a convergence condition of the iterative procedure. It is

$$R_{i,j} \leq 2\Delta t^{n+1/2} R_r / S_{i,j} \cdot \Delta x_i \cdot \Delta y_j \qquad (13)$$

If it is desired that all $R_{i,j}$ be equal (for convenience in construction), then their values should be selected such that

$$R_{i,j} \leq \{2[\min(\Delta t^{n+1/2})]R_r\}/[\max(S_{i,j}\cdot\Delta x_i\cdot\Delta y_j)] \qquad (14)$$

The ratio (i.e., scale) V_r enables one to convert voltages measured in the resistor network into water levels for the aquifer. Information transfer from the network to the digital computer (the distribution of V) and back (the distribution of U) can be performed automatically if the two are linked by suitable equipment and made into a single hybrid computer.

The hybrid computer. The principle components of the hybrid computer are shown in Figure 3.

The roles of the various components may be summarized as follows. The roles of the resistor network are representation of the aquifer's shape, representation of the aquifer's boundary conditions, representation of the transmissivities, and solution of a set of simultaneous linear equations for each iteration, within each time step. The roles of the digital computer are control and management of the computation process, representation of the storativity, representation of the initial conditions, representation of inputs (recharge) and outputs (pumping) for each time interval, computation of the input voltages U to be supplied to the network, and transformation of voltages to water levels and vice versa. The roles of the intermediate equipment are conversion of signals from analog to digital and from digital to analog; setting of voltage values for

ϕ = 0.0m

	0.238	0.047	0.015	0.047	0.238	
	0.277	0.060	0.021	0.060	0.277	
ϕ = 1.0m	0.282	0.062	0.022	0.062	0.282	ϕ = 1.0m
	0.277	0.060	0.021	0.060	0.277	
	0.238	0.047	0.015	0.047	0.238	

ϕ = 0.0m

Fig. 4. Model shape, boundary conditions, and final heads.

TABLE 1. Results for the (Simulated) Hybrid Computer

Time Interval	Iterations	Maximum Difference in Water Levels, mm
1	5	0.5
2	4	0.4
3	4	0.4

the network, as determined by computations in the digital computer; and sampling of voltages in the network.

The hybrid computer as described above may be used as a general device for solving five-diagonal equations of the same form as (4). This was done by *Vemuri and Karplus* [1969] for identifying model parameters. *Brandt* [1969] suggests a binary method for fixing the input voltages U, which simplifies the intermediate equipment.

Comparison of the Methods

For the purpose of this investigation we have simulated the hybrid computer by a program on the digital computer. The program was constructed so that it enabled us to examine the performance of each component, without having actually to construct and assemble expensive equipment. The items examined included the number of iterations, the times needed for each task, and their comparison to a pure numerical solution. The following is one of the examples which were used.

The model employed is shown in Figure 4, where the aquifer shape, boundary conditions, and final heads are indicated. Other data are $\Delta x_i = \Delta x = 1000$ m, $\Delta y_j = \Delta y = 1000$ m, $T_{i,j} = T = 1000$ m²/d, $S_{i,j} = S = 0.2$, $\Delta t^n = \Delta t = 30$ days, $p_{i,j}{}^n = 0$, and $r_{i,j}{}^n = 0$. Initial conditions are $\phi_{i,j}{}^0 = 0$ m. The computations were performed for $N = 3$ time steps. For the resistor network the following data were used: $R_r = 10^6\ \Omega$ m m/d, $V_r = 1$ V/m, and $R_{i,j} = 10\ \Omega$.

Table 1 summarizes results obtained for the (simulated) hybrid computer: the number of iterations per time step and the maximum difference between values of ϕ in the last two iterations (where the tolerance allowed was 1 mm).

Table 2 shows the split of computation time between the simulation of the resistor network and the digital tasks. Some 75% of the total time was spent on the simulation. Not all this time could be saved if the hybrid computer were indeed constructed, since this time included computation of the voltages U and some time would be needed for sampling the voltages V and setting up the voltages U. A more meaningful comparison is between the time needed for the pure digital tasks in the simulation of the hybrid computer and the total computation time by the ADI method, as shown in Table 3.

Thus even if all other tasks of the hybrid solution are instantaneous, the ADI numerical solution is far superior, at least for this size model. This should be examined for larger problems, a difficult task because the simulation program requires much computer memory and considerable computation time. At any rate, since using a hybrid computer would also entail a considerable construction cost, there is little to warrant it as a substitute for a straight numerical solution.

TABLE 2. Split of Computation Time for Simulation and for Digital Tasks

Time Interval	Time for Digital Computations, %	Time for Simulation, %
1	24.5	75.5
2	25.4	74.6
3	28.0	72.0

TABLE 3. Comparison of Computation Time (Seconds) for the ADI Method and the Digital Tasks of the Hybrid Solution

Time Interval	Numerical Solution by ADI	Digital Tasks of the Hybrid Solution
1	0.09	0.22
2	0.04	0.18
3	0.04	0.16

Discussion and Conclusions

Finite difference models for forecasting water levels in aquifers have been presented by *Pinder and Bredehoeft* [1968], *Prickett and Lonnquist* [1971], *Blank* [1971], and the authors [*Hefez et al.*, 1972]. The last three works include the computer programs and instructions for their use.

We have examined the noniterative ADI method and have determined the size of memory required and computation time (on two computers) as a function of the number of cells in the model. The results are shown in Figure 3. An investigation was also carried out on the performance of the semihybrid computer proposed by *Karplus* [1964, 1965]. This investigation, performed with the use of a program simulating the semihybrid computer, shows that the pure numerical solution is superior from all points of view, at least for models of the size investigated. Further work in this area is required for larger models.

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