Optimal Design and Operation of Water Distribution Systems

URI SHAMIR[1]

IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598

A methodology is developed for optimal design and/or operation of a water distribution system that is to operate under one or several loading conditions. Decision variables may be design variables, such as pipe diameters, or control variables, such as heads and flows. The objective function may include the initial cost of the design, the cost of operation, the physical measures of performance, and the penalties for violating constraints. Constraints may be imposed on the decision variables and on the performance of the system under each loading. Flow solutions are obtained by a modified Newton-Raphson method employing sparse matrix techniques. Optimization is obtained by a combination of the generalized reduced gradient and penalty methods. Implementation in a computer program and its use on a test problem in both batch and time-sharing modes are described, and it is concluded that the method is computationally feasible. The many different ways in which it can be used to analyze, design, and operate water distribution systems are outlined.

Water distribution systems are interconnected networks of closed conduits, valves of various types, pumps, and reservoirs. The model of a network is made of links connected at nodes. With each node we associate a head, which is a measure of the hydraulic energy at the node, and a consumption, which is the quantity of water withdrawn from the network (or injected into it) at the node. Withdrawals in real networks are distributed along pipelines, and the consumptions at nodes of the model represent the aggregate of these withdrawals over an appropriate area. Associated with each link is a resistance law, which relates the flow through the link to the head loss (or gain, as is the case for pumps) between the ends of the link. In this law there is a numerical coefficient, called the resistance, which depends on the physical properties of the link (e.g., length, diameter, and roughness for a pipe). Certain types of links, such as pumps, may require more than one coefficient in their resistance law; for simplicity of presentation we shall ignore such cases. Reservoirs are connected to the system at certain nodes. For each reservoir a relation is given between the volume stored and the water level; from it one can compute the changes in level due to flows into or out of the reservoir. The water level in a reservoir is also the head at the node to which it is connected.

We focus on looped networks, which are the common case in urban systems. The methodology to be presented is applicable to branching networks (tree shaped, with no loops), but more efficient methods exist for such networks that take advantage of their simple hydraulic solution.

This paper presents one aspect of work aimed at providing tools for analysis, design, and operation of water distribution systems. Some of the problems faced by designers, analysts, and engineers are to design a new system, to construct and calibrate a mathematical model of an existing system, to analyze the performance of an existing system with the aim of designing improvements and modifications, and to operate an existing system. Often the task may be merely to find an acceptable, or feasible, solution to one or more of the above problems, that is, to find a solution that satisfies certain conditions without regard to achieving any goal beyond feasibility. For such purposes one would like to have tools, and by tools we mean computer programs, which can efficiently analyze the behavior of the system. The task would then be accomplished by successive analyses of alternative solutions until an acceptable one is found. The efficiency of programs used in this mode depends not only on the computation time needed to perform one analysis but also to a great extent on the kind of information that they generate and the ease with which one can make modifications to the data and rerun the program. The usefulness of the results in advancing to the next alternative to be examined will determine the total number of runs to be made and the man and machine time that will have to be spent in arriving at the final solution. This paper will allude to these aspects of the work only briefly. We concentrate here on situations where the word optimize is added in any one of the tasks mentioned above. In such cases one has in addition to the constraints, which define what the feasible alternatives are, an objective function, which is a measure of the desirability of each alternative.

Thus within the overall framework of the development of tools for analysis, design, and operation of water distribution systems this paper concentrates on the optimization and touches on the other aspects only to the degree necessary to develop the optimization procedure and to put it in perspective in the general scheme of the work.

REVIEW OF PREVIOUS WORK

Past work on optimization of water distribution systems has concentrated on the design problem; i.e., the decision variables are design parameters such as pipe diameters, pump capacities, etc. The objective function commonly used has been the sum of two parts: the initial cost and the present value of operating costs for one or more representative operating conditions. Each operating condition is a steady state hydraulic solution of the network, i.e., a situation in which heads, consumptions, and flows remain constant. This represents one loading condition (also referred to simply as loading) that is given by a set of demands to be supplied and the values of the fixed heads in

[1] On leave from the Department of Civil Engineering, Technion-Israel Institute of Technology, Haifa, Israel.

the system (such as those at some or all reservoirs). The loading in real systems constantly changes with the time of day, the day of the week, and the season. In optimization models, one uses some representative or typical loadings, for example, the maximum hourly demands to be expected during a year, the average hourly demands of the peak day, the average hourly demands over a year, etc. Each loading is assumed to prevail during some portion of the time during the design horizon. The attendant variable costs of operating the system under each loading, most of which are the costs of energy for pumping, are a function of the design parameters. These costs are converted into present value and are added to the initial cost to yield the value of the objective function of the design alternative.

In an early work on the optimization of the design of water distribution systems [*Shamir*, 1964] the decision variables were the diameters of all pipes in a system to be designed. The objective function considered one loading condition, and the operating cost was related to the energy lost in the flow through all the pipes. The consumptions were all given for the one loading, and no constraints were imposed on the heads or flows. The optimization was performed by a gradientlike technique. Components of the gradient of the objective function were computed, and the pipe whose diameter, when it was changed, would give the largest improvement in the objective function was changed to the next commercially available value. The flow solution, i.e., obtaining the steady state hydraulic solution (see next section), was obtained by a Newton-Raphson method, and the Jacobian of this solution was then used in computing the components of the gradient. The work presented herein can be viewed as a descendant of this early work, which was presented also by *Lemieux* [1965]. The method for solving the flow problem was subsequently generalized [*Shamir and Howard*, 1968], and a sensitivity analysis was added that also used the Jacobian of the flow solution. These features will be explained in later sections. (*Stoner* [1972] has applied the same method of solution and the sensitivity analysis to natural gas networks.)

Pitchai [1966] used a random sampling technique to search for the optimal diameters of a pipe network operating under a number of loadings. The objective function contained the initial cost and one (the most costly) loading. The flow solution was obtained by the Newton-Raphson method, but no use was made of the Jacobian in seeking the direction for improving the design. Constraints on heads were taken into consideration by adding penalties on constraint violation to the objective function to be minimized. *Smith* [1966] combined random search, steepest descent, and linear programing to optimize the design in a formulation similar to Pitchai's.

Jacoby [1968] treated the same problem. He used a numerical gradient technique; i.e., an approximate gradient of the objective function was computed by making small moves around the current design point and evaluating the function at these points. Two more types of moves were also used: a move in a direction selected at random and an 'experience' move determined by the relative success of several previous moves. Diameters were considered continuous variables, and the values obtained in the optimization were rounded to the nearest commercially available size. This rounding may cause the design so obtained to be infeasible, and some adjustment may be necessary.

Karmeli et al. [1968], *Gupta* [1969], and *Gupta et al.* [1972] dealt with optimal design of branching networks. Once the consumptions are given, the flow in each link of the branching network is known, and therefore the effect of changing the resistance of links on heads at nodes can be computed directly (i.e., with no iterations). Since the resistance of a pipe and its cost are linear functions of its length (but not of its diameter), if one selects pipe lengths as decision variables, the optimization can be cast as a linear program. Thus the decision variables were taken as the lengths of the segments of pipes with given diameters in each link, where the set of diameters for each link is selected in advance. Constraints on heads at nodes appear as linear inequalities imposed on the decision variables. Since the selection of the admissible set of diameters for each link is done merely for computational feasibility (unless there is some real constraint on the available diameters), one may have to change the sets of admissible diameters and re-solve the problem until the implicit constraint introduced by specifying the admissible diameters is not binding (i.e., the solution does not contain a link made entirely of one diameter that is at the limit of its admissible set). These works considered only the initial cost in the objective function, but it is not difficult to include operating costs.

Kally [1972] extended the method to looped networks, using the same decision variables (length of diameter in link) and objective function (initial cost only). In looped networks the effect of changes in these decision variables on the heads at the nodes cannot be computed directly. Kally used an approximate method as follows: each of the decision variables in turn is changed by 1 unit (i.e., 1 unit of length is added to it), the new network is solved, and the changes in all heads recorded. These are then used as the coefficients in the linear inequalities, even though the changes in heads are only approximately linear for changes in the decision variables. The network obtained as a solution of the linear program is again solved, new coefficients for head changes are computed, and a new linear program is formulated. The process is repeated until it converges, i.e., until the solution of the linear program does not change on successive iterations.

Kohlhaas and Mattern [1971] used separable programing to determine the optimal diameters, pumps, and reservoirs in a looped system in which all heads were given in advance. Linear constraints are obtained in this case if flows are used as decision variables, and since the heads are known, one can easily determine the diameters once the flows have been fixed. The objective function is nonlinear in the flow decision variables and contains the cost of pipes, pumps, and reservoirs.

Deb and Sarkar [1971] used a special formulation, called the equivalent diameter concept, to determine the optimal diameters in a network once the pressure surface profile (i.e., the heads) and the head at the inlet are known. They then developed a method for determining the optimal pressure surface profile and the inlet head. The objective function includes the initial cost and reflects the cost of one loading condition through the cost of the reservoir and pumps required to supply the prescribed demands.

In a systems analysis of proposed additions to the primary water network of New York, *de Neufville et al.* [1971] did not solve an optimization problem. They formulated several performance indices to evaluate each alternative: the weighted average of the residual pressures above a minimum fixed standard at several key points in the system (where the weighting factor is the quantity supplied at each point), the residual pressure at the extreme end of the system, and a measure of the degradation in the first performance index (weighted residual pressure) that would result from outage of one major link in the network. Alternative designs were selected by the analysts. Each was solved hydraulically, and the cost and performance indices for it computed. All alternatives were displayed on plots for each performance index versus the cost, and these data were then used in the final selection process. This work relied on results obtained by *Lai and Schaake* [1969], where linear programing was used to find the least costly network satisfying given constraints. The formulation as a linear program was based on two steps: variable transformation, which linearized the constraints, and a linear approximation of the objective function. The objective function considered one loading condition, although the theoretical development was extended to multiple loadings.

A rapid development has occurred in recent years in optimizing the operation of power systems. Newton's method (the same as the Newton-Raphson method) was adopted for solving the power flow problem, and it became the accepted method after the development of methods for taking advantage of the sparsity of the Jacobian [*Tinney and Walker*, 1967]. Sensitivity analysis was added [*Peschon et al.*, 1968], and a method for obtaining optimal power flow solutions was developed [*Dommel and Tinney*, 1968]. The methodology presented in this paper draws on the experience gained in solving increasingly complex optimization problems in power system operation [*Dommel*, 1969, 1972; *Peschon et al.*, 1972*a*, *b*].

We have not found any previous work on optimization of the operation of water systems besides a special purpose procedure developed by *Dreizin et al.* [1971], and all the work summarized above has dealt with the design. In power systems, on the other hand, the work has concentrated on optimal operation. The work reported herein has combined the two.

Since the formulation and the solution of the optimization problem use the flow solution, we devote the following sections to the flow solution and to some of its extensions.

Flow Solutions

Mathematical models. It is an accepted and justified practice to use a schematic representation of the real system in the mathematical model. The degree of schematization should depend on the problem being solved. Past work, however, has tended to use arbitrary schematization rules, such as ignoring all pipes below some chosen size, and it is proposed that substantial benefits can often be derived by more extensive schematization, i.e., by reducing the number of elements in the model, without adversely affecting the quality of the results. A discussion of this point, which would be complete only with specific examples and the like, is beyond the scope of this paper, and can be found in a report by the author [*Shamir*, 1973]. Here we merely accept the mathematical model as a representation of the real system. The representation of the steady state flow problem in a water distribution system is a set of simultaneous nonlinear algebraic equations. There are several ways in which the model can be set up, the most common of which are (1) node equations, i.e., a continuity (conservation of mass) equation for each node, or (2) loop equations, i.e., continuity of the hydraulic grade line (the energy line) around each of a selected set of closed loops.

The model for the flow problem used in this work is based on node equations. Consider a network made of NL links connected at N nodes. When Q_{ji} denotes the flow into node j from node i (which is free to be positive or negative) and C_j denotes the external flow into node j (the consumption, taken as being positive when it enters the node), the set of continuity equations for all nodes is

$$\sum_{i=1}^{N} Q_{ji} + C_j = 0 \qquad j = 1, \cdots, N \tag{1}$$

The flow Q_{ji} is a function of the heads at nodes i and j and of the resistance R_{ij} of the link between them. Obviously, $Q_{ji} \equiv 0$ when there is no link between nodes i and j. Thus (1) can be written

$$\sum_{i=1}^{N} f_{ij}(H_i, H_j, R_{ij}) + C_j = 0 \qquad j = 1, \cdots, N \tag{2}$$

There is a total of $2N + NL$ variables: a head and a consumption at each node and a resistance for each link. To be able to solve (2), which is the mathematical model of the network, exactly N of the variables can be unknown. The unknowns may be heads, consumptions, and resistances, but their combination must be such that (2) is solvable [*Shamir and Howard*, 1968; *Shamir*, 1973]. Denote by $x = (x_1, \cdots, x_N)$ the vector of unknowns (dependent or basic variables) and by y the vector of all remaining $N + NL$ variables. Then (2) can be written

$$\begin{aligned} &G_j(x, y) = 0 \qquad j = 1, \cdots, N \\ \text{or} \qquad &[G(x, y)] = \mathbf{0} \end{aligned} \tag{3}$$

Method of solution. A modified Newton-Raphson technique is used to solve (3). At iteration k the values of the unknown are $x^k = (x_1^k, \cdots, x_N^k)$. The residuals at this point are

$$G_j^k = G_j(x^k, y) \qquad j = 1, \cdots, N \tag{4}$$

A check is performed for convergence. We use as a criterion for terminating the iterations

$$\max_j \{|G_j^k|\} \leq EPS \tag{5}$$

where EPS is the maximum acceptable error in satisfying the continuity equation at any node.

If (5) is not satisfied, a new iteration is started. A vector Δx^k is computed from the set of simultaneous linear equations

$$[J^k]\cdot[\Delta x^k] = [-G^k] \tag{6}$$

where J^k is the Jacobian

$$[J] = \begin{bmatrix} \partial G_1/\partial x_1 & \cdots & \partial G_1/\partial x_N \\ \vdots & & \vdots \\ \partial G_N/\partial x_1 & \cdots & \partial G_N/\partial x_N \end{bmatrix} \tag{7}$$

evaluated at some appropriate value of x. One possibility would be

$$J^k = J(x^k) \tag{8}$$

that is, to compute J always at the current point x^k, but this is not necessarily the most efficient way. Since there is really no meaning to the 'accuracy' of Δx^k obtained from (6), it has been found to improve overall efficiency if J is recomputed only every few iterations. The frequency of recomputation can also be decreased as the solution progresses and the changes in x between successive iterations become smaller.

At this point some consideration should be given to the method for solving (6), which is a key issue in the efficiency of the flow solution and also of the optimization presented later. Typically, each node is connected to no more than, say, four of five other nodes. Therefore there are only in that order nonzero elements per row of J (note that the number of nonzero entries in a row equals the number of unknowns appearing in that node's equation, which depends on but need not be equal to the number of nodes to which it is connected); J is thus sparse; for $N = 100$ it may have of the order of 5% nonzero entries. This immediately indicates the desirability of using spare matrix techniques in solving (6) to save both storage space and computation time. Such techniques have been developed and used very effectively in solving power flow problems by Newton's method. (See *Tinney and Meyer* [1972] for a review; see *Tinney and Walker* [1967], *Ogbuobiri et al.* [1970], and *Shamir* [1973] for technical details.) The method used is called ordered triangular factorization, in which the Jacobian is factorized into a lower triangular matrix L and a unit upper triangular matrix U:

$$[J] = [L]\cdot[U] \tag{9}$$

The factorization, which in effect amounts to the operations performed in a Gaussian elimination, is carried out in a compact storage scheme. In this scheme, only nonzero entries of the matrices are stored. Additional vectors contain the positions of these entries in the matrices and information on the sequence of operations performed in the factorization. Once J has been factorized, the solution of (6) is obtained in two steps:

$$[L][z] = [-G^k] \tag{10}$$

is solved for z by forward substitution; then, by means of z from (10),

$$[U]\cdot[\Delta x^k] = [z] \tag{11}$$

is solved for Δx^k by backward substitution.

The method results in substantial savings in computation time, since only nontrivial operations are performed. The pivoting strategy in the factorization is aimed at retaining sparsity, i.e., at reducing the amount of fill-in that occurs during the elimination process. Once J has been factorized, it can be used very efficiently to solve with new right-hand-side vectors in (6), hence the motivation not to recompute the Jacobian on every iteration. The same factorized matrix is also used in solving several other equations that arise in the sensitivity analysis and optimization presented later, where the matrix is either J or its transpose.

To return to the iterative solution of (3), once Δx^k has been computed, the new values of the unknowns are obtained from

$$x_i^{k+1} = x_i^k + \alpha_i^k \, \Delta x_i^k \qquad i = 1, \cdots, N \tag{12}$$

The multipliers α are obtained by observing the progress of the solution over a few past iterations and using heuristic rules to guide the solution to a quicker convergence. With the new values, one returns to computing the residuals, as defined by (4).

The above presentation lacks detail. A full discussion of all aspects mentioned can be found in a report by *Shamir* [1973].

SENSITIVITY ANALYSIS

Once (3) has been solved, one can investigate the sensitivity of the solution to variations in the values of the known variables y. If a variable y_m is perturbed, since (3) has to hold for the new value, then

$$dG_j/dy_m = 0 \qquad j = 1, \cdots, N \tag{13}$$

The total derivative is expanded:

$$\frac{dG_j}{dy_m} = \frac{\partial G_j}{\partial y_m} + \sum_{i=1}^{N} \frac{\partial G_j}{\partial x_i}\frac{\partial x_i}{\partial y_m} = 0 \quad j = 1, \cdots, N \tag{14}$$

and the resulting equation can be written in matrix form as

$$\left[\frac{\partial G}{\partial x}\right]\cdot\left[\frac{\partial x}{\partial y_m}\right] = \left[-\frac{\partial G}{\partial y_m}\right] \tag{15}$$

where $[\partial x/\partial y_m]$ is the vector of sensitivities of the N dependent variables to changes in y_m; $[\partial G/\partial x]$ is the Jacobian, evaluated at the solution point of (3), i.e., the last Jacobian of the flow solution; and $[-\partial G/\partial y_m]$ is a vector of terms computed in the same way as those of the Jacobian and is also sparse. For each variable y_m to be considered, one calculates a new right-hand-side vector in (15) and solves with the already factorized Jacobian, which, as was stated above, requires very little computation.

The sensitivity analysis is useful in calibrating the mathematical model of an existing system and in studying proposed modifications in the design or operation of a system [*Shamir and Howard*, 1968; *Peschon et al.*, 1968; *Stoner*, 1972; *Shamir*, 1973]. It also appears as an implicit component in the optimization.

FORMULATION OF THE OPTIMIZATION MODEL

The methodology to be presented in this and the following section is applicable to a wide range of problems. It enables the solution of problems in both design and operation. It can also be used in several other ways that will be explained later. The general formulation considers the cost of the design and the cost of operating the system under one or several loadings. Some of the terms may be missing from the objective function, depending on the problem to be solved, and other special purpose terms may be added, as will be explained later, to deal with cases where the solution sought is optimal in a sense other than economic.

Decision variables. Denote by $d = (d_1, \cdots, d_{m_0})$ the vector of design variables. These are the decision variables associated with design, such as pipe diameters, pump capacities, etc. The design may encompass an entire new system or may include only modifications and additions to an existing system. Denote by $u^l = (u_1^l, \cdots, u_{m_l}^l)$ the vector of control variables associated with the lth loading. These are the decision variables for the operation of the system under the lth loading, such as heads at various points of the system, quantities to be supplied, valve settings, pumps on/off commands, etc. If a total of L loadings is being considered, then $u = (u^1, \cdots, u^l, \cdots, u^L)$ is a vector of vectors, each of which may be a different size and may include different control variables; (d, u) is the vector of all decision variables.

During the optimization, when each flow problem is being solved, the values of the decision variables are (temporarily) fixed. Thus (d, u^l) is part of the vector y^l of known variables of the lth flow problem. We denote by s^l the vector of fixed variables that are neither decision variables nor dependent variables in the lth flow problem. Thus $y^l = (d, u^l, s^l)$. The vector of dependent variables of the lth flow problem is denoted x^l.

Objective function. The general form of the objective function is

$$F(d, u, x, s) = f(d) + \sum_{l=1}^{L} w^l c^l(d, u^l, x^l, s^l) \qquad (16)$$

where $f(d)$ is the initial cost of the design, c^l is the cost of operating the system for 1 time unit (say, an hour) under the lth loading, and w^l is a weighting factor; c^l depends implicitly on the design variables d and the fixed variables s^l, since they determine the values in x^l; c^l depends explicitly on the control variables u^l and the dependent variables x^l, since they may include quantities of water to be pumped and heads affecting the pumping costs; w^l reflects several factors: a present value factor for annual costs, the number of time units (say, hours) per year that the lth loading is supposed to prevail, and possibly a relative importance factor assigned subjectively by the analyst to deal with special cases.

For a design problem, one includes the initial cost term and the weighted costs of a few, say, one to a maximum to five, loading conditions. For a control problem, i.e., a problem of operation, one drops the initial cost term and includes the cost of operation with only one loading (the weight can then obviously be made unity).

Constraints. There are three types of constraints:

1. The decision variables have to be chosen from an admissible set of values.
2. The flow problems have to be satisfied.
3. The dependent variables of the flow problems have to be within permissible ranges.

Optimization problem. The optimization problem may be expressed as

$$\min_{d,u} F(d, u, x, s) = f(d) + \sum_{l=1}^{L} w^l c^l(d, u^l, x^l, s^l) \qquad (17)$$

subject to

$$d \in D \qquad (18)$$

$$u^l \in U^l \qquad \forall l \quad (19)$$

$$[G^l(d, u^l, x^l, s^l)] = \mathbf{0} \qquad \forall l \quad (20)$$

$$x^l = \{x \mid [G^l(d, u^l, x, s^l)] = \mathbf{0}\} \in X^l \qquad \forall l \quad (21)$$

Some or all of the design variables may be required to take on only certain integer values, as is the case when they are diameters of commercially available pipes. Most of the control variables may be continuous (heads, consumptions, pipe resistances affected by valve settings, etc.) or may be treated as being continuous even though they are discrete (e.g., take as the control variable the pump's outlet head in lieu of an on/off control variable for its operation). All these cases can be handled by the optimization technique. On/off control variables pose difficulties the remedies for which will be investigated further.

Common forms for (18) and (19) are

$$d \min_p \leq d_p \leq d \max_p \qquad (22)$$

and

$$u \min_q{}^l \leq u_q{}^l \leq u \max_q{}^l \qquad (23)$$

an additional requirement being that pipe diameters be even integers.

Equation 20 represents L separate flow problems, i.e., L sets of N simultaneous nonlinear equations, each set to be solved separately. Equation 21, which actually includes (20), represents the constraints on the dependent variables of the flow problems and usually has the simple form

$$x \min_i{}^l \leq x_i{}^l \leq x \max_i{}^l \qquad (24)$$

METHOD OF SOLUTION

The solution is based on the generalized reduced gradient method [*Abadie*, 1970] and draws on some of its variants used in optimal power flow solutions [*Dommel and Tinney*, 1968; *Dommel*, 1972; *Peschon et al.*, 1972*a*, *b*; *Velghe and Peterson*, 1972].

A Lagrangian is formed by using the objective function and the flow equations (20):

$$\mathcal{L}(d, u, x, s, \lambda) = F(d, u, x, s) + \sum_{l=1}^{L} \sum_{i=1}^{N} \lambda_i{}^l G_i{}^l(d, u^l, x^l, s^l) \qquad (25)$$

which can be written as

$$\mathcal{L} = F + [G]^T \cdot [\lambda] \qquad (26)$$

where T is the sign for transpose. Now at any point (d, u) the following has to hold:

$$\left[\frac{\partial \mathcal{L}}{\partial x}\right] = \mathbf{0} = \left[\frac{\partial F}{\partial x}\right] + \left[\frac{\partial G}{\partial x}\right]^T [\lambda] \qquad (27)$$

Since the G and x can be separated into the independent flow problems, (27) can be decomposed into

$$\left[\frac{\partial \mathcal{L}}{\partial x^l}\right] = \mathbf{0} = \left[\frac{\partial F}{\partial x^l}\right] + \left[\frac{\partial G^l}{\partial x^l}\right]^T [\lambda^l] \qquad (28)$$

$$l = 1, \cdots, L$$

Thus the Lagrange multipliers are made of L groups, N values being in each group, which are the solution of

$$[J^l]^T[\lambda^l] = [-\partial F/\partial x^l] \qquad l = 1, \cdots, L \qquad (29)$$

which is similar to (6) or (15) except that the transpose of the Jacobian appears. Once the Jacobian has been factorized, solving (29) involves the same computational effort as one repeat solution of (6) or (15) with a new right-hand side and is therefore very efficient.

Once the Lagrange multipliers are available, the reduced gradient of F is computed from

$$[\nabla F] = \begin{bmatrix} \nabla\ d \\ \\ \nabla\ u \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathcal{L}}{\partial d} \\ \\ \dfrac{\partial \mathcal{L}}{\partial u} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F}{\partial d} \\ \\ \dfrac{\partial F}{\partial u} \end{bmatrix} + \left[\frac{\partial G}{\partial d} ; \frac{\partial G}{\partial u}\right]^T [\lambda] \qquad (30)$$

In expanded form the components are given by

$$\nabla\ d_p = \frac{\partial F}{\partial d_p} + \sum_{l=1}^{L} \sum_{j=1}^{N} \left(\frac{\partial G_j{}^l}{\partial d_p}\right) \lambda_j{}^l \qquad (31)$$

$$p = 1, \cdots, m_0$$

and

$$\nabla u_q{}^l = \frac{\partial F}{\partial u_q{}^l} + \sum_{j=1}^{N} \left(\frac{\partial G_j{}^l}{\partial u_q{}^l}\right) \lambda_j{}^l \qquad (32)$$

$$q = 1, \cdots, m_l$$

At a (local) minimum, $[\nabla F] = \mathbf{0}$. At any other point, the vector $[-\nabla F]$ points in the direction of steepest descent of F when the changes in the flow solutions due to a move in this direction are taken into account. (Note that the vector $[\partial F/\partial d; \partial F/\partial u]$ does not by itself satisfy this condition.) When the constraints on the design and control variables as given in (22) and (23) are taken into consideration, a direction for a move is given by the vector

$$\begin{aligned} r_p &= 0 \qquad \nabla\ d_p > 0 \qquad d_p = d \min{}_p \\ r_p &= 0 \qquad \nabla\ d_p < 0 \qquad d_p = d \max{}_p \\ r_p &= -\nabla\ d_p \qquad \text{otherwise} \end{aligned} \qquad (33)$$

and similarly for each $u_q{}^l$. Expression 33 is the reduced gradient projected into the feasible decision space. A move is now made to a new point in the decision space given by

$$\begin{aligned} d_p(\text{new}) &= d_p(\text{old}) + \Delta d_p = d_p(\text{old}) + \beta_p r_p \\ u_q{}^l(\text{new}) &= u_q{}^l(\text{old}) + \Delta u_q{}^l = u_q{}^l(\text{old}) + \beta_q{}^l r_q{}^l \end{aligned} \qquad (34)$$

where the multipliers β are yet to be determined. If all β are equal, the move is in the direction of the steepest feasible descent. When θ denotes the value of the β for this case, the question remains how to select the best value of θ. We have used the following method:

1. Denote as point 1 (the base) the point at which the value of F and the direction vector $[r]$ have just been computed.

2. Select a trial value for θ, dependent on the 'size' of the space of the decision variables (the 'diagonal' of the hypercube defined by equations 22 and 23). Using this value of θ, compute from (34) the location of point 2, and compute the value of F there.

3. Using the values of the objective function at points 1 and 2 and the slope of the objective function at point 1 (the derivative of F in the direction of $[r]$), fit a parabola that is an approximation of the change of F along $[r]$, and find the point at which it reaches its minimum. This point is called point 3, and it becomes point 1 for the next move.

The success of this procedure depends on several parameters selected heuristically and on the way that the parabola is fitted and used. Further experimentation is being carried out to improve this one-dimensional search, and other methods (e.g., Fibonacci) will also be tried. Furthermore, if one allows the β in (34) to be different, one can achieve moves in improved directions, based either on simple observations of the relative success of a few previous moves (similar to the way the α were selected for equation 12) or on the method of conjugate gradients [*Fletcher and Reeves*, 1964].

To aid in the one-dimensional search, one can compute an approximation to the changes in the dependent variables that would result from the move. Denote by $(\Delta d, \Delta u)$ the (trial) move in the direction $[r]$. Since the equations of each flow problem will still have to be satisfied after the move, one can equate the total derivative of each G^l to zero and obtain

$$\left[\frac{\partial G^l}{\partial x^l}\right][\Delta x^l] + \left[\frac{\partial G^l}{\partial d}\right][\Delta d] + \left[\frac{\partial G^l}{\partial u^l}\right][\Delta u^l] = 0 \qquad (35)$$

$$l = 1, \cdots, L$$

where, again, $[\partial G^l/\partial x^l] = [J^l]$ is the Jacobian of the last iteration of the lth flow problem. The other matrices have also been computed before (equations 30, 31, and 32), and so solving for Δx^l is again a single repeat solution with the factorized Jacobian. Now the best value of θ, θ^* is found from the one-dimensional search:

$$\min_{0 \le \theta \le \bar{\theta}} F[d + \theta\ \Delta d,\ u + \theta\ \Delta u,\ x + \theta\ \Delta x] \qquad (36)$$

The maximum step that can be taken is given by $\bar{\theta}$, which is the maximum θ such that the constraints

$$\begin{aligned} d \min{}_p &\le d_p + \theta\ \Delta d_p \le d \max{}_p \\ u \min{}_q{}^l &\le u_q{}^l + \theta\ \Delta u_q{}^l \le u \max{}_q{}^l \\ x \min{}_i{}^l &\le x_i{}^l + \theta\ \Delta x_i{}^l \le x \max{}_i{}^l \end{aligned} \qquad (37)$$

are still satisfied. The point thus reached can be designated as point 2 for the one-dimensional search indicated by (36).

There are two ways in which the constraints (24) on the dependent variables can be dealt with: penalty methods and variable exchange. In the first way, use is made of penalty functions, which are added to the objective function [*Carroll*, 1961; *Dommel and Tinney*, 1968, p. 1870; *Peschon et al.*, 1972*a*, p. 67]. We have used

$$\begin{aligned} P(x_i{}^l) &= p_i{}^l(x \min{}_i{}^l - x_i{}^l)^2 \qquad x_i{}^l < x \min{}_i{}^l \\ P(x_i{}^l) &= p_i{}^l(x_i{}^l - x \max{}_i{}^l)^2 \qquad x_i{}^l > x \max{}_i{}^l \\ P(x_i{}^l) &= 0 \qquad \text{otherwise} \end{aligned} \qquad (38)$$

where the multipliers $p_i{}^l$ are set initially to a relatively small value and increased on successive iterations of the optimization so as to force the final solution to satisfy (24) exactly or at least to some acceptable tolerance. Incidentally, there may be a reason to allow (24) to be satisfied only approximately, as is the case when pressures are allowed to drop somewhat below the standard in certain parts of the system, if this helps improve the overall operation of the system. The initial (relatively small) values assigned to the penalty multipliers, their increase on successive iterations, and their maximum allowed values are all controlled by the user of the program and require some degree of experience. Large values of the multipliers cause sharp and narrow valleys to be formed in the response surface (objective function plus penalties); such valleys hinder the progress of the optimization, since it proceeds along a sequence of straight-line segments, and one needs a large number of relatively short segments to follow a narrow valley. The penalties should therefore not be increased too rapidly. Alternative formulations for penalty functions will be investigated, such as the formulation proposed by *Powell* [1969], which is reported to have improved computational properties. When penalties are added to the objective function, it is the combined function that is used wherever F or its derivatives appear in the equations above.

The other method for dealing with constraints on the dependent variables of the form given by (24) is to exchange variables between the vectors x and y. Whenever a dependent variable reaches one of its bounds, it is fixed at that bound for the next iteration. At the same time, one of the previously fixed variables is released and is made into a dependent variable. The exchange of variables must be such that the flow problems are solvable with the new division of variables into known and dependent. Also, the released variable should indeed be one that can be made free to change in the real system. The exchange is performed after the move indicated by (36) has been determined. If θ^* is such that one or more of the x reach the bound, as is determined in (37), then these x have to be fixed at that bound and exchanged with some y [*Abadie*, 1970; *Velghe and Peterson*, 1972].

To deal with integer requirements on decision variables, one can either make every move in the search such that these requirements are met or proceed for a while as if they did not exist and only toward the end of the search impose this condition. The first method has been used in this study, but more experimentation is needed to determine which method is better.

The search is terminated when any one of the following conditions is met: (1) $[r] \cong 0$ (to within some specified tolerance), indicating either a (local) minimum or a boundary point that cannot be improved; (2) a small move indicated in (36); and (3) the exceeding of a specified number of iterations.

IMPLEMENTATION

The optimization procedure outlined above, using the penalty functions given in (38) and employing the parabola fit for the one-dimensional search, has been implemented in a computer program. The program is essentially an extension of the network solver, which obtains solutions to flow problems with the modified Newton-Raphson method, (12) being used as the correction formula and the Jacobian being updated only after several iterations. Sparse matrix techniques are used in factorizing the Jacobian and in solving the equations where it or its transpose appears. The use of the sparsity techniques is a major factor in determining the overall efficiency of the optimization and its feasibility with respect to computer storage.

The program is made of a number of subroutines. Most are general purpose. Those defining the specific form of the objective function and its derivatives have to be supplied by the user. The main program merely defines a sequence of tasks, and so by modifying it the analyst can perform a desired sequence of tasks, such as obtaining one or several flow solutions, performing sensitivity analyses, or optimizing the design and/or operation.

The program can handle networks with up to 100 nodes and 180 links, can consider 5 loadings, and can deal with 50 design variables and 10 control variables for each of the 5 loadings. For this it requires approximately 179,200 bytes of memory.

EXAMPLE

The 25-node 40-pipe network shown in Figure 1 has been used to test the optimization procedure. This is a synthetic network whose data (pipe lengths and diameters, heads and consumptions) were selected at random over some specified range and are given in Table 1. The consumptions shown in the table are assumed to represent an average loading condition that holds 50% of the time during the design horizon. Another loading, whose consumptions are 1.5 times those in Table 1, is assumed to occur 10% of the time. Under both loadings the unknowns in the flow problems are the consumptions at the four corner nodes and the heads at the 21 other nodes. The fixed heads at the four corner nodes have for both loadings the values shown in Table 1. For variables that are unknown in the flow problems, Table 1 gives the initial values for the solution.

Two design variables were considered: the diameters of pipes 1 and 21. Two control variables were considered: the head at node 1 for each of the loadings. The decision variables are

$$d = (d_1, d_2) = [\mathrm{diam}(1), \mathrm{diam}(21)]$$

$$u^1 = (u_1^{\,1}) = (H_1^{\,1})$$

$$u^2 = (u_1^{\,2}) = (H_1^{\,2})$$

The initial cost part of the objective function is

$$\sum 0.5 * L * \mathrm{diam}^{1.3} = 0.5 * L(1) * \mathrm{diam}(1)^{1.3} + 0.5 * L(21) * \mathrm{diam}(21)^{1.3} = 0.5[1825 * \mathrm{diam}(1)^{1.3} + 1055 * \mathrm{diam}(21)^{1.3}]$$

Operating costs are assumed to be related to pumping certain quantities of water into and out of specified nodes against the head difference between the node and some fixed external head. The total operating cost is

$$\sum_{l=1,2} w^l \sum_{i=1,5,20} C_i^{\,l}\, \Delta H_i^{\,l}$$

The coefficients w^l are computed as follows: (unit weight of water, 62.4 lb/ft^3) $\times$ (cost of energy, \$1.85 $\times$ 10^{-9}/lb ft, which is 5 mils/kWh) $\times$ (present value factor, 28.4 for 2.5% over 50 yr) $\times$ (number of seconds per year, 31.5 $\times$

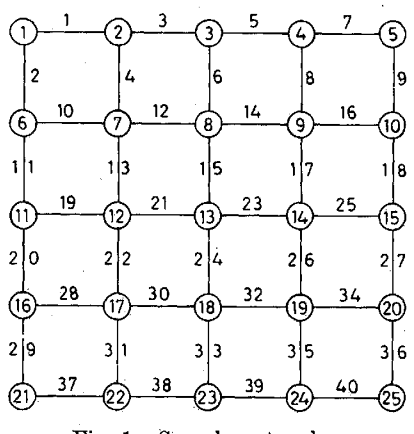

Fig. 1. Sample network.

TABLE 1. Input Data for the Sample Network

Pipe No.	Length, feet	Diameter, in.	Node	Head, feet	Consumption, gpm
1	1825	28	1	146.45	22,244
2	1544	34	2	140.65	−16,469
3	1843	16	3	148.33	24,250
4	1998	26	4	141.52	−32,523
5	1671	36	5	145.22	6,735
6	1711	22	6	141.40	−31,694
7	1169	24	7	145.93	24,443
8	1900	28	8	143.19	−31,493
9	1950	22	9	148.09	21,730
10	1628	32	10	143.94	−6,622
11	1173	24	11	148.22	16,459
12	1521	36	12	148.89	11,742
13	1884	24	13	144.47	13,125
14	1432	32	14	149.14	28,817
15	1266	28	15	148.64	2,859
16	1128	18	16	146.11	−6,102
17	1963	24	17	145.15	10,305
18	1933	18	18	140.23	−46,894
19	1557	24	19	143.45	−6,426
20	1532	36	20	140.27	−14,650
21	1055	18	21	140.97	−9,671
22	1759	22	22	147.30	10,646
23	1763	20	23	143.56	11,000
24	1529	36	24	143.26	−10,702
25	1691	32	25	147.90	9,939
26	1169	36			
27	1599	18			
28	1621	16			
29	1593	20			
30	1007	34			
31	1207	20			
32	1892	22			
33	1807	36			
34	1606	32			
35	1438	24			
36	1089	16			
37	1284	22			
38	1205	20			
39	1039	30			
40	1100	26			

10^6) = \$103 s/ft^4. The result is multiplied by 0.5 for the first loading and by 0.1 for the second; i.e., $w^1 = 51.5$ \$ s/ft^4; $w^2 = 10.3$ \$ s/ft^4.

The consumptions to be pumped and the heads to or from which they are to be delivered are: C_1^l, pumped from a head of 50 feet into node 1; C_5^l, pumped from a head of 50 feet into node 5; and C_{20}^l, pumped to a head of 250 feet from node 20.

Thus the specific form of the second part of the objective function is

$$51.5[C_1^1(H_1^1 - 50) + C_5^1(H_5^1 - 50) + C_{20}^1(H_{20}^1 - 250)] + 10.3[C_1^2(H_1^2 - 50) + C_5^2(H_5^2 - 50) + C_{20}^2(H_{20}^1 - 250)]$$

These variables appear in the flow problems as

Known

$$H_1^l,\ H_5^l,\ C_{20}^l$$

Unknown

$$C_1^l,\ C_5^l,\ H_{20}^l$$

Of these, the H_1^l, $l = 1, 2$, are decision variables.

At the initial design point the data are diam(1) = 28 in., diam(21) = 18 in., and $H_1^1 = H_1^2 = 146.45$ feet. The initial cost part for these data is \$92,027. Table 2 includes the data for computing the operating cost part at the initial design point.

TABLE 2. Data for Computing the Operating Cost Part at the Initial Design Point

	$l = 1$					$l = 2$				
	C, gpm	C, ft^3/s	H, feet	ΔH, feet	$C\Delta H$	C, gpm	C, ft^3/s	H, feet	ΔH, feet	$C\Delta H$
Node 1	21,110	48.0	146.45	96.45	4360	30,790	68.5	146.45	96.45	6,700
Node 5	6,611	14.8	145.22	95.22	1410	10,250	22.9	145.22	95.22	2,175
Node 20	−14,650	−32.7	141.58	−108.42	3650	−21,970	−48.8	136.33	−113.67	5,530
Total					9420					14,405

The operating cost for these data is $51.5 \times 9420 + 10.3 \times 14{,}405 \cong \$636{,}000$ (the program actually computed \$635,757). The total cost for the initial design point is \$727,784, about 13% of which is initial cost.

The constraints on the decision variables were: diam(1), equal to even integers between 10 and 40 in.; diam(21), equal to even integers between 10 and 30 in.; and $H_1{}^l$, equal to continuous variables between 138 and 150 feet.

The constraints on the dependent variables were that all heads were to be above 138 feet. No upper bounds were given on the heads, nor any limitations on the consumptions (the four that were dependent variables).

The initial value given to the penalty multipliers was 10, and they were multiplied by a factor of 10 on successive iterations. Since the number of iterations was small, the penalties did not grow to an appreciable percent of the objective function. For this reason the final solution was quite far from satisfying the constraints on the dependent variables. Most heads were very close to 138 feet or above it, but one was as low as 121.25 feet (at node 2 for the second loading). Higher initial values of the penalties were used, and the result was improved.

For the problem posed, the optimum was found at a corner point of the feasible design region: diam(1) = 10 in., diam(21) = 10 in., and $H_1^1 = H_1^2 = 138$ in. The final value of the objective function was \$588,320, of which \$28,732, or 5%, was the initial cost. The overall improvement from \$727,784 to \$588,320 should not be viewed as being typical in any sense, since the example is a purely hypothetical one.

The optimum was reached in a small number of iterations, always under 10. The same optimum was reached from several starting points. The whole solution took less than 10 s (execution only) on the 360/91.

The same program has also been implemented and run under the time-sharing system on an IBM 360/67. In this mode the self-guiding search of the optimization procedure has been supplemented by user-controlled moves. Data are displayed on the terminal, showing the current point, the value of the objective function and its reduced gradient, the penalty functions, the move to be made, etc. The analyst can then decide either to let the program proceed to the next step in the optimization or first to modify some of the data (various parameters, penalty multipliers, the next point to be tried, etc.) and only then to proceed. He can thus investigate a few alternative designs around the optimum, can test the effect of various terms, and can in general gain a lot of insight into the behavior of the network with respect to the objective function and other aspects.

POSSIBLE APPLICATIONS

As was stated in the introductory section, the aim of the overall work has been to provide tools for analysis, design, and operation of water distribution networks. The methodology for optimization presented in this paper and the way in which it has been implemented in the computer program should be viewed in this general context. This becomes clear when one considers the many different problems that can be tackled with the use of the program:

1. Solve a flow problem.
2. Perform sensitivity analyses of a flow solution.
3. Simulate the behavior of a system over time when it has storage (reservoirs). Steady state solutions, each representing some period of time (say, one to several hours) during which conditions do not change much, are linked via the changes in water levels in reservoirs. The levels are assumed to remain constant for the duration of the time period, at the end of which the levels are updated by converting the flows into or out of the reservoirs into changes in levels. Simulation over time is important in studying the role of storage in balancing the loads on the sources and on the network.
4. Find a feasible design, given one or several loadings and constraints on the decision and dependent variables. The objective function would contain only penalties for violating constraints on the dependent variables. Once a feasible design is reached, the value of the objective function is zero, and any design satisfying this condition is acceptable.
5. Find designs that are optimal with respect to physical functions, such as the weighted residual pressure criterion used by *de Neufville et al.* [1970].
6. Optimize the design of a new system or of additions to an existing one, disregarding costs of operation. The decision variables are all design parameters, and the objective function contains only the initial cost.
7. Optimize the operation of an existing system for one loading. Decision variables may be heads, consumptions, and also link resistances (representing valve settings). The objective function is the cost of operating under the given loading for 1 unit of time.
8. Optimize the design and operation for several loadings. This is the general problem dealt with in this paper.
9. Calibrate the mathematical model of an existing system. The decision variables are usually link resistances (say, the roughness coefficients of some or all pipes) but may in certain cases also include heads or consumptions. The objective function to be minimized is some measure of the difference (say, the sum of the squares of the deviations) between the computed values of some or all dependent variables for one or several loadings and the measured values of the same variables. The optimization would then yield the model's parameter values that give the best fit. Note, by the way, that a good calibration should be based on more than one loading, preferably loadings that are quite different in magnitude and pattern.

Acknowledgments. The support of IBM Israel, IBM World Trade Corporation, and IBM Research Division, which enabled my stay at the Watson Research Center, is gratefully acknowledged.

REFERENCES

Abadie, J., Application of the GRG method to optimal control problems, in *Integer and Nonlinear Programming*, edited by J. Abadie, chap. 8, pp. 191–211, North-Holland, Amsterdam, 1970.

Carroll, C. W., The created response surface technique for optimizing nonlinear restrained systems, *Oper. Res.*, *9*, 169–185, 1961.

Deb, A. K., and A. K. Sarkar, Optimization in design of hydraulic networks, *J. Sanit. Eng. Div. Amer. Soc. Civil Eng.*, *97*(SA2), 141–159, 1971.

de Neufville, R., J. C. Schaake, and J. H. Stafford, Systems analysis of water distribution networks, *J. Sanit. Eng. Div. Amer. Soc. Civil Eng.*, *97*(SA6), 825–842, 1971.

Dommel, H. W., Constrained optimal control of real and reactive power dispatch, paper presented at Symposium on Optimal Power System Operation, Univ. of Manchester, Manchester, England, Sept. 1969.

Dommel, H. W., Optimization of power systems, in *Techniques of Optimization,* edited by A. V. Balakrishnan, pp. 487–498, Academic, New York, 1972.

Dommel, H. W., and W. F. Tinney, Optimal power flow solutions, *IEEE Trans. Power App. Syst., PAS-87*(10), 1866–1876, 1968.

Dreizin, Y., N. Arad, U. Shamir, and H. Matmon, Evaluation of savings in energy in regional water systems (in Hebrew), *Rep. 131.105,* 85 pp., Res. and Desalination Div., Mekorot Water Co., Israel, April 1971.

Fletcher, R., and C. M. Reeves, Function minimization by conjugate gradients, *Computer J., 7,* 149–154, 1964.

Gupta, I., Linear programming analysis of a water supply system, *Trans. Amer. Inst. Ind. Eng., 1*(1), 56–61, 1969.

Gupta, I., M. Z. Hassan, and J. Cook, Linear programming analysis of a water supply system with multiple supply points, *Trans. Amer. Inst. Ind. Eng., 4*(3), 200–204, 1972.

Jacoby, S. L. S., Design of optimal hydraulic networks, *J. Hydraul. Div. Amer. Soc. Civil Eng., 94*(HY3), 641–661, 1968.

Kally, E., Computerized planning of the least cost water distribution network, *Water Sewage Works,* 121–127, 1972.

Karmeli, D., Y. Gadish, and S. Meyers, Design of optimal water distribution networks, *J. Pipeline Div. Amer. Soc. Civil Eng., 94*(PL1), 1–10, 1968.

Kohlhaas, C., and D. E. Mattern, An algorithm for obtaining optimal looped pipe distribution networks, in *Papers of the 6th Annual Symposium on the Application of Computers to the Problems of the Urban Society,* pp. 138–151, Association of Computing Machinery, New York, 1971.

Lai, D., and J. C. Schaake, Linear programming and dynamic programming application to water distribution network design, part 3, Engineering Systems Analysis of the Primary Water Distribution Network of New York City, 111 pp., Dep. of Civil Eng., Mass. Inst. of Technol., Cambridge, July 1969.

Lemieux, P. F., Minimum cost design of water pipe networks, M.S. thesis, Dep. of Civil Eng., Mass. Inst. of Technol., Cambridge, 1965.

Ogbuobiri, E. C., W. F. Tinney, and J. W. Walker, Sparsity directed decomposition for Gaussian elimination on matrices, *IEEE Trans. Power App. Syst., PAS-89*(1), 141–150, 1970.

Peschon, J., D. S. Piercy, W. F. Tinney, and O. J. Tviet, Sensitivity in power systems, *IEEE Trans. Power App. Syst., PAS-87*(8), 1687–1696, 1968.

Peschon, J., D. W. Bree, and L. P. Hajdu, Optimal power-flow solutions for power system planning, *Proc. IEEE, 60*(1), 64–70, 1972*a*.

Peschon, J., H. W. Dommel, W. Powell, and D. W. Bree, Optimum power flow for systems with area interchange controls, *IEEE Trans. Power App. Syst., PAS-91*(3), 898–905, 1972*b*.

Pitchai, R., A model for designing water distribution pipe networks, Ph.D. thesis, Harvard Univ., Cambridge, Mass., 1966.

Powell, M. J. D., A method for nonlinear constraints in minimization problems, in *Optimization-Symposium of the Institute of Mathematics and Its Applications,* edited by R. Fletcher, pp. 283–297, Academic, New York, 1969.

Shamir, U., Minimum cost design of water distribution networks, Dep. of Civil Eng., Mass. Inst. of Technol., Cambridge, 1964.

Shamir, U., Water distribution systems analysis, *Rep. RC 4389,* 290 pp., IBM Thomas J. Watson Res. Center, Yorktown Heights, N. Y., 1973.

Shamir, U., and C. D. D. Howard, Water distribution systems analysis, *J. Hydraul. Div. Amer. Soc. Civil Eng., 94* (HY1), 219–234, 1968.

Smith, D. V., Minimum cost design of linearly restrained water distribution networks, M.S. thesis, Dep. of Civil Eng., Mass. Inst. of Technol., Cambridge, 1966.

Stoner, M. A., Sensitivity analysis applied to a steady state model of natural gas transportation systems, *Soc. Petrol. Eng. J., 12*(2), 115–125, 1972.

Tinney, W. F., and W. S. Meyer, Solution of large sparse systems by ordered triangular factorization, paper presented at Joint Automatic Control Conference, Stanford Univ., Stanford, Calif., Aug. 1972.

Tinney, W. F., and J. W. Walker, Direct solutions of sparse network equations by optimally ordered triangular factorization, *Proc. IEEE, 55*(11), 1801–1809, 1967.

Velghe, J., and N. M. Peterson, Optimal control of real and reactive power flow under constraints, paper presented at Power System Computation Conference, Grenoble, France, Sept. 1972.

(Received April 25, 1973;
revised September 18, 1973.)