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WATER DISTRIBUTION SYSTEMS ANALYSIS

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INTRODUCTION

A network of pipes and hydraulic elements (valves, pumps, reservoirs) is considered solved when the heads and consumptions at all nodes in the network are known. Obtaining the solution, as defined herein, consists of finding the values of the specified unknowns which satisfy the following physical laws of the network: (1) Preservation of mass continuity at each node; and (2) that for each element there is a known relationship between discharge and energy gradient.

The Hazen-Williams equation, commonly used for water distribution studies, was selected as the law relating pipe discharge to energy loss. Other equivalent equations can be selected if desired.

Whether for the analysis of an existing network or for the design of a new one, the engineer needs the capability to solve for various combinations of unknowns, under many loading conditions. The analytical tools which have been developed to date make this task a lengthy and tedious process. The present work takes full advantage of the Newton-Raphson method to solve directly for combinations of unknowns which may include heads, consumptions, and element resistances. The method incorporates pumps, valves and other elements into the method of solution, without recourse to special external procedures.

The generalized steady-state solution of a water distribution network is but a small part of an over-all system analysis. Additional aspects such as acquisition, processing, storing, and retrieval of data, control and operation of a

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network, economic and social implications, and the effective use of digital computers will be described in another paper.

REVIEW OF PREVIOUS WORK

The oldest method for systematic solution of distribution networks, and the one still most commonly used, is the Hardy Cross method.[3] This method is well suited for solution by hand, and is easily adapted for machine computation. Computer programs written to perform the Hardy Cross analysis are described by Hoag and Weinberg,[4] Graves and Branscome,[5] Adams,[6] and Dillingham.[7]

Electronic network analyzers are discussed by McIlroy,[8] McPherson and Radziul,[9] and others. A summary of methods and techniques was presented by McPherson.[10]

Other methods of mathematical solution of networks resulted from work on electrical networks. Warga[11] applied Duffin's[12] work on nonlinear networks to distribution networks. Warga proved the existence of a unique solution for the heads at the nodes under steady state flows in a network whose elements (an element is a link connecting two nodes) satisfy certain conditions. If the general law relating flow and head loss in any element is given by

$$Q_{ji} = f_{ji}\,(H_i - H_j) \quad \text{(1)}$$

then these conditions are that: (1) $f_{ji}(x) = -f_{ij}(-x)$; (2) $f_{ji}(x)$ is continuous for all x; (3) for all couples (j, i), $f_{ji}(x)$ is nondecreasing as x increases; (4) there exists a path between any two nodes in the network along which every element has an $f_{ji}(x)$ which increases as x increases and takes on all values (i.e., $-\infty < f_{ji} < \infty$).

Warga discusses[11] two iterative procedures for solving a network which satisfies the above conditions. One procedure always converges, although slowly, from any starting assumption. The other procedure, the Newton-Raphson technique, converges rapidly from a reasonable assumption, but may

[3] Cross, Hardy, "Analysis of Flow in Networks of Conduits or Conductors," Bulletin No. 286, Univ. of Illinois Engrg. Experimental Station, Urbana, Ill., 1936.

[4] Hoag, L. N., and Weinberg, G., "Pipeline Network Analysis by Electronic Digital Computer," Journal of the American Water Works Association, Vol. 49, 1957, pp. 517-524.

[5] Graves, Q. B., and Branscome, D., "Digital Computers for Pipeline Network Analysis," Journal of the Sanitary Engineering Division, ASCE, Vol. 84, No. SA2, April, 1958, pp. 1608-1—1608-18.

[6] Adams, R. W., "Distribution Analysis by Electronic Computer," Journal of the Institute of Water Engineers, Vol. 15, 1961, pp. 415-428.

[7] Dillingham, J. H., "Computer Analysis of Water Distribution Systems, Part II," Water and Sewage Works, Feb., 1967, pp. 43-45.

[8] McIlroy, M. S., "Direct Reading Electric Analyzer for Pipeline Networks," Journal of the American Water Works Association, Vol. 42, Apr., 1950, p. 347.

[9] McPherson, M. B., and Radzidul, J. V., "Water Distribution Design and the McIlroy Network Analyzer," Journal of the Hydraulics Division, ASCE, Vol. 84, No. HY2, Apr., 1958, pp. 1588-1—1588-15.

[10] McPherson, M. B., "Application of System Analyzers—A Summary," Water and Sewage Works, Reference Number, 1962, pp. R53-R67.

[11] Warga, J., "Determination of Steady State Flows and Currents in a Network," Proceedings, Instrument Society of America, Vol. 9, Pt. 5, 1954, Paper 54-43-4.

[12] Duffin, R. J., "Nonlinear Networks," Bulletin of the American Mathematical Society, Vol. 53, 1947, pp. 963-971.

not converge at all if the initial assumption is unreasonable.

The existence and uniqueness of the solution of a network has not been considered for cases when the unknowns include consumptions at nodes and element resistances. The problem of existence and uniqueness is even more complicated when elements such as pumps, for which condition (3) above does not hold, are included in the network.

Martin and Peters[13] used the Newton-Raphson method in a computer program to solve for the unknown heads at the nodes of a network of pipes (no pumps or valves). They reported no difficulty with convergence.

Shamir[14] used the same method as one of a set of computer programs for economic analysis of water distribution networks. These programs, which were subsequently used by Lemieux[15] and Smith,[16] solved for unknown heads, and convergence was always achieved. Giudice[17] added a sensitivity analysis and considered the case of more than one fixed head.

Pitchai[18] used the Newton-Raphson method and treated networks with pumps as boundary conditions. The conditions at the pumps were satisfied outside the general network solution (the pumps were not included directly in the analysis as elements in the network). Pitchai does not present proof of the convergence of the method, nor of the existence of a solution for a network including pumps. He stresses, however, that when a solution is known to exist (based on engineering judgment) the method converges rapidly.

In summary, previous studies of pipeline networks have extensively treated the problem of solving for heads at the nodes, but little has been done to solve for other types of unknowns. Studies were restricted to consideration of networks of pipes and included other elements (such as pumps) by special procedures which were external to the basic method of network solution.

SOLUTION OF A PIPELINE NETWORK

Consider a network of NJ nodes and NL pipes. The statement of continuity for all nodes is

$$F_j = \sum_{i=1}^{NJ} Q_{ji} + C_j = 0 \qquad j = 1, \ldots, NJ \qquad (2)$$

[13] Martin, D. W., and Peters, G., "The Application of Newton's Method to Network Analysis by Digital Computer," Journal of the Institute of Water Engineers, Vol. 17, 1963, pp. 115-129.

[14] Shamir, U., "Minimum Cost Design of Water Distribution Networks," Report, Dept. of Civil Engineering, M. I. T., Cambridge, Mass., 1964, (unpublished).

[15] Lemieux, P. F., "Minimum Cost Design of Water-Pipe Networks," thesis presented to Massachusetts Institute of Technology, at Cambridge, Mass., in 1965, in partial fulfillment of the requirements for the degree of Master of Science.

[16] Smith, D. V., "Minimum Cost Design of Linearly Restrained Water Distribution Networks," thesis presented to Massachusetts Institute of Technology, at Cambridge, Mass., in 1966, in partial fulfillment of the requirements for the degree of Master of Science.

[17] Giudice, J. J., "Analysis of Pipe Networks Based on the Newton-Raphson Method," thesis presented to Massachusetts Institute of Technology, at Cambridge, Mass., in 1965, in partial fulfillment of the requirements for the degree of Master of Science.

[18] Pitchai, R., "A Model for Designing Water Distribution Pipe Networks," thesis presented to Harvard University, at Cambridge, Mass., in 1966, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

in which Q_{ji} = the discharge from i to j (Q_{ji} = 0 when no pipe connects nodes j and i), and C_j = the consumption at node j (C is positive when it is an input to the node).

The Hazen-Williams equation can be written

$$Q = 6.2 \times 10^{-4}\ C_{HW}\ D^{2.63} \left(\frac{\Delta H}{L}\right)^{0.54} \quad \ldots\ldots\ldots\ldots \quad (3)$$

in which Q = discharge, in cfs; C_{HW} = the Hazen-Williams coefficient of the pipe; D = pipe diameter, in in.; ΔH = head loss along the pipe, in ft; and L = pipe length, in ft.

Eq. 3 is used in the form

$$Q_{ji} = \frac{H_i - H_j}{R_{ij}^{0.54}\ |H_i - H_j|^{0.46}} \quad \ldots\ldots\ldots\ldots \quad (4)$$

in which R_{ij}, the resistance of the pipe connecting nodes i and j, is given by

$$R_{ij} = \frac{850260\ L_{ij}}{C_{HW_{ij}}^{1.85}\ D_{ij}^{4.87}} \quad \ldots\ldots\ldots\ldots \quad (5)$$

Eq. 4 is written in a form which guarantees a consistent sign convention for discharge (i.e., $Q_{ji} > 0$ means flow from i to j). Eq. 4 can now be used to write Eq. 2 (the continuity equation) in terms of heads and consumptions at nodes and pipe resistances, as follows

$$F_j = \sum_{i=1}^{NJ} \frac{H_i - H_j}{R_{ij}^{0.54}\ |H_i - H_j|^{0.46}} + C_j = 0 \qquad j = 1, \ldots, NJ \quad \ldots\ldots \quad (6)$$

The elements in the summation of Eq. 6 are non zero only if nodes i and j are connected by a network element. This will be implied in all subsequent summations.

For a network, if $F_j = 0$ is satisfied at all nodes, then the external balance equation

$$\sum_{j=1}^{NJ} C_j = 0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

is satisfied.

Since there are NJ simultaneous equations one can solve for NJ unknowns. They may be heads, consumptions, or resistances. For the network to be solvable the combination of unknowns must satisfy the conditions outlined subsequently. As the equations are non linear, the solution is achieved by successive iterations, using a suitable method which achieves convergence. The chosen method of solution, the Newton-Raphson technique, finds a new set of improvements or corrections to the values of the unknowns in each iteration. The improvements are computed from the first term of a Taylor expansion about the present state of the solution.

The Newton-Raphson method may be conveniently illustrated for the one-dimensional case shown in Fig. 1 as follows. The value x_o is sought, such that

$$f(x)\big|_{x = x_o} = f(x_o) = 0 \quad \ldots\ldots\ldots\ldots \quad (8)$$

At the k^{th} iteration the approximation for x_o is denoted by x_k. The next approximation is given by

$$x_{k+1} = x_k + \Delta x_k = x_k - \frac{f(x_k)}{\frac{df(x_k)}{dx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

in which $df(x_k)/dx$ is the derivative of $f(x)$ evaluated at x_k. The equation for the k^{th} improvement Δx_k, can then be written

$$f(x) + \frac{df}{dx} \Delta x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which it is understood that both $f(x)$ and its derivative are evaluated using the present value of x.

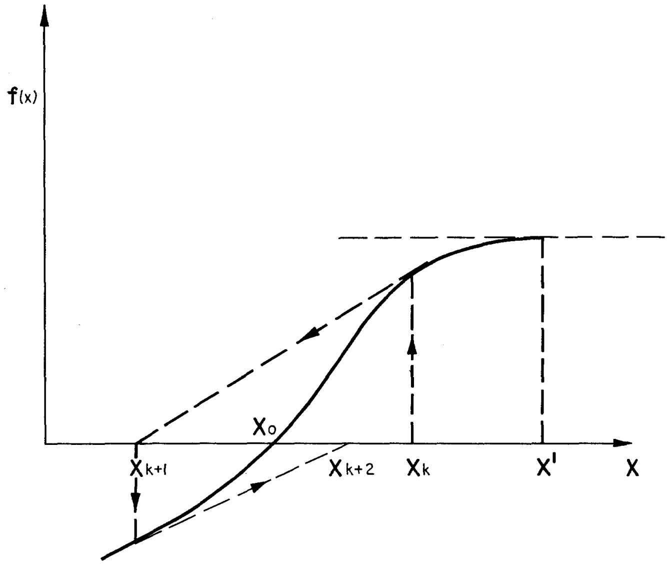

FIG. 1.—ILLUSTRATION OF THE NEWTON-RAPHSON METHOD FOR THE ONE-DIMENSIONAL CASE

When there are n equations to be satisfied $[f_1(x_1, \ldots, x_n) = 0, \ldots, f_n(x_1, \ldots, x_n) = 0]$ and n unknowns $(x_1, \ldots, x_n)$ to be solved for, the set of n improvements $(\Delta x_1, \ldots, \Delta x_n)$ are the solution of the set of n simultaneous linear equations

$$f_j(x_1, \ldots, x_n) + \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \Delta x_i = 0 \qquad j = 1, \ldots, n \quad \ldots\ldots (11)$$

Consider now the network having NJ nodes. The set of unknown heads is denoted by $\bar{H}$, the set of unknown consumptions by $\bar{C}$, and the set of unknown resistances by $\bar{R}$. The NJ simultaneous equations for the corrections are

$$F_j(\bar{R}, \bar{H}, \bar{C}) + \sum_{R_{ij} \in \bar{R}} \frac{\partial F_j}{\partial R_{ij}} \Delta R_{ij} + \sum_{H_i \in \bar{H}} \frac{\partial F_j}{\partial H_i} \Delta H_i$$

$$+ \sum_{C_i \in \bar{C}} \delta_{ij} \Delta C_i = 0 \quad j = 1, \ldots, NJ \qquad (12)$$

in which $H_i \in \bar{H}$ signifies "H_i is in the set $\bar{H}$" (i.e., H_i is an unknown), and δ_{ij} is the Kroneker delta ($\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise). At each iteration the set of equations given by Eq. 12 may be solved for the corrections ΔR_{ij} ($R_{ij} \in \bar{R}$), ΔH_i ($H_i \in \bar{H}$) and ΔC_i ($C_i \in \bar{C}$). These corrections are then added algebraically to the present values of the unknowns to obtain a new estimate of the solution. A check is then made to determine if all equations given by Eq. 6 are satisfied to within some specified error criterion. This error criterion is the amount by which any F_j may be different from zero, and represents the maximum allowable unbalanced discharge at any node. If, in checking the present solution, the error criterion is not met at any of the nodes, a new iteration is begun. The magnitude of the error criterion has a major influence on the number of iterations which will be required to reach the accepted solution.

The partial derivatives in Eq. 12, obtained from Eq. 6, are given by

$$\frac{\partial F_j}{\partial H_i} = \frac{0.54}{R_{ij}^{0.54} \, |H_i - H_j|^{0.46}} = \frac{\partial F_i}{\partial H_j} \qquad (13)$$

$$\frac{\partial F_j}{\partial H_j} = - \sum_{i \neq j} \frac{\partial F_j}{\partial H_i} \qquad (14)$$

and

$$\frac{\partial F_j}{\partial R_{ij}} = \frac{-0.54\,(H_i - H_j)}{R_{ij}^{1.54} \, |H_i - H_j|^{0.46}} \qquad (15)$$

Eq. 12 can be written in matrix form

$$\begin{bmatrix} \cdots \frac{\partial F_1}{\partial R_{pq}} \cdots\cdots \frac{\partial F_1}{\partial H_s} \cdots\cdots \frac{\partial F_1}{\partial C_t} \cdots \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \\ \cdots \frac{\partial F_{NJ}}{\partial R_{pq}} \cdots\cdots \frac{\partial F_{NJ}}{\partial H_s} \cdots\cdots \frac{\partial F_{NJ}}{\partial C_t} \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \Delta R_{pg} \\ \vdots \\ \Delta H_s \\ \vdots \\ \Delta C_t \\ \vdots \end{bmatrix} = \begin{bmatrix} -F_1 \\ \vdots \\ -F_{NJ} \end{bmatrix} \qquad (16)$$

The matrix of derivatives on the left side is called the Jacobian of the set of equations. The numerical values in this matrix and on the right side vary from one iteration to the next, as they are computed at each iteration with a new set of values for the unknowns. The solution of Eq. 16 was readily obtained for each iteration by using the Gauss-Jordan elimination procedure.

To illustrate the structure of Eq. 16, it is written for the network shown in Fig. 2, a sample network, with $NJ = 5$ and $NL = 7$.

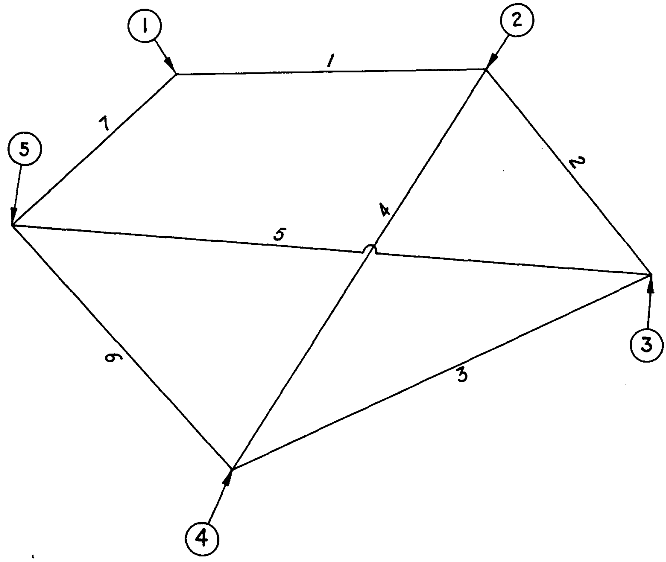

FIG. 2.—SAMPLE NETWORK

The unknowns for this example are

$$\bar{H} = (H_1, H_3) \qquad (17a)$$

$$\bar{C} = (C_3, C_4) \qquad (17b)$$

and $$\bar{R} = (R_{24}) = (R_4) \qquad (17c)$$

and Eq. 16 takes the form

$$\begin{bmatrix} 0 & \frac{\partial F_1}{H_1} & 0 & 0 & 0 \\ \frac{\partial F_2}{\partial R_4} & \frac{\partial F_2}{\partial H_1} & \frac{\partial F_2}{\partial H_3} & 0 & 0 \\ 0 & 0 & \frac{\partial F_3}{\partial H_3} & 1 & 0 \\ \frac{\partial F_4}{\partial R_4} & 0 & \frac{\partial F_4}{\partial H_3} & 0 & 1 \\ 0 & \frac{\partial F_5}{\partial H_1} & \frac{\partial F_5}{\partial H_3} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta R_4 \\ \Delta H_1 \\ \Delta H_3 \\ \Delta C_3 \\ \Delta C_4 \end{bmatrix} = \begin{bmatrix} -F_1 \\ -F_2 \\ -F_3 \\ -F_4 \\ -F_5 \end{bmatrix} \qquad (18)$$

in which the derivatives are given by Eqs. 13, 14, and 15.

Since the continuity equations involve only differences between heads, it is necessary to specify at least one head to establish a datum. If the problem is to solve for all the heads in the network, given all the consumptions and resistances, then it is necessary, to fix one head, consider one consumption or resistance as unknown, and solve for it together with the $(NJ - 1)$ unknown heads.

SYSTEM ELEMENTS OTHER THAN PIPES

In addition to pipes, a water distribution system contains a variety of elements. These may be pumps, valves, elevated tanks, hydrants, and other types of control and measuring devices. It is essential to be able to incorporate the special characteristics of these elements into the analysis, and to be able to solve the system as it physically exists. The method presented herein is ideally suited to perform such an analysis.

Consider the case for which the link connecting nodes k and l of a network is an element other than a pipe, such as a pump or valve. Then for the link kl,

$$Q_{kl} = f_E\,(H_k - H_l) = -Q_{lk} \quad \ldots \quad (19)$$

in which f_E is a given function. Eq. 19 may be the characteristic curve of a pump, a loss curve for a valve, etc. In Eq. 16, the derivatives, with respect to H_k and H_l, are given by

$$\frac{\partial F_k}{\partial H_k} = -\sum_{\substack{j \neq k \\ j \neq l}} \frac{0.54}{R_{kj}^{0.54}\,\left|H_k - H_j\right|^{0.46}} + \frac{\partial f_E}{\partial H_k} \quad \ldots \quad (20)$$

$$\frac{\partial F_k}{\partial H_l} = \frac{\partial f_E}{\partial H_l} \quad \ldots \quad (21)$$

$$\frac{\partial F_l}{\partial H_k} = -\frac{\partial f_E}{\partial H_k} \quad \ldots \quad (22)$$

and

$$\frac{\partial F_l}{\partial H_l} = -\sum_{\substack{j \neq l \\ j \neq k}} \frac{0.54}{R_{lj}^{0.54}\,\left|H_l - H_j\right|^{0.46}} - \frac{\partial f_E}{\partial H_l} \quad \ldots \quad (23)$$

all other derivatives in Eq. 16 remaining the same (given by Eqs. 13, 14, and 15). The value of f_E and its derivatives can be computed from measured data taken from field tests of the element being considered. As an example, consider the network shown in Fig. 2. We now replace the pipe connecting nodes 3 and 5 by a pump for which

$$Q_{35} = f_E = Q_o - \alpha\,(H_3 - H_5)^\beta \qquad H_3 > H_5 \quad \ldots \quad (24a)$$

and

$$Q_{35} = f_E = 0 \qquad H_3 < H_5 \quad \ldots \quad (24b)$$

Eq. 24 implies that the pump operates only as long as the head at node 3 is higher than that at node 5, i.e., it always pumps from node 5 to node 3. For this case the terms $\partial F_3/\partial H_3$; $\partial F_3/\partial H_5$; $\partial F_5/\partial H_3$ and $\partial F_5/\partial H_5$ are computed from Eqs. 20–23, using f_E from Eq. 24. All other terms in the matrix of derivatives are the same as in Eq. 18.

Eq. 24 can be used to illustrate the difficulties which may arise when elements other than pipes are considered. In general, the characteristic equation for each type of element considered (pump, valve, etc.) is different. Although the function f_E in Eq. 24 is continuous, its derivatives are not. This means that when F_3, F_5 or their derivatives with respect to H_3 or H_5 are computed at any iteration of the solution, a check must be made to determine whether H_3 is larger or smaller than H_5. A decision must then be made to use either Eq. 24*a* or Eq. 24*b*—whichever applies. The convergence of the iterative scheme can not be guaranteed when any of the characteristic functions of elements in the network do not have continuous derivatives. This will be considered subsequently.

Any type of element may be included in the network analysis as long as its characteristic curve is available. A pipe can be considered as just one of many types of elements, its characteristic curve being given by the Hazen-Williams equation.

A set of programs to perform the analysis was developed and implemented on a digital computer in a time sharing environment.

APPLICATIONS

The present method of analysis has broad application to water distribution systems analysis. Pumps, valves, hydrant-pumper combinations, or other network elements with known head-discharge characteristics, can be incorporated directly into the solution, without recourse to external ad hoc procedures. Alternatively, it is possible to calculate a required element characteristic for specified system performance. For example, it is possible to compute directly the pipe resistances required to produce specified pressures and consumptions. Trial and error procedures are thereby eliminated.

Of major importance is the high degree of flexibility afforded in setting up the data for a run or changing the basic network layout for subsequent runs. Nodes or elements can be added or deleted from the basic data file without a major reshuffle.

The location and types of unknowns can be freely interchanged, from run to run, without any changes in the data or ordering of the computations. Consequently, the method is extremely well suited to real time use for decision making as part of a man-machine team. It is also well suited for use in the control and operation of an existing network. This last application is a specialized topic and will be discussed in another paper.

SOLVABILITY OF NETWORKS

To demonstrate the fact that one cannot solve for any set of NJ unknowns without exception in a network of NJ nodes, consider the extremely unfavorable case for which the NJ unknowns (which include heads, consumptions, and resistances) are all concentrated in one region of the network. It should be intuitively obvious that even if the flows and heads at the pipes coming into this area are all known, it is impossible to solve this part of the network. Thus, the solvability of a network depends on the way in which the NJ unknowns are distributed.

There do not seem to be any general rigorous rules for determining whether a network which includes all types of unknowns is, or is not, solvable. There is, however, one simple rule which follows immediately from the method of solution used, and is apparent from the examination of Eq. 16. Eq. 16 has a unique solution for the NJ unknowns if, and only if, the rank of the matrix of coefficients is NJ. This will obviously not be the case if any row of the matrix contains only zeros. A row of zeros would result if there were no unknowns appearing in the continuity equation of some node in the network. To avoid this, one has to distribute the unknowns throughout the network in such a way that there should be at least one unknown appearing in the continuity equation of each node. Considering any node, at least one of the following should be unknown: (1) The consumption at the node; (2) the heads at the node itself or at adjacent nodes; or (3) the resistance of a pipe which connects to the node. Observing this rule will eliminate the most common reason for making a network unsolvable. This does not imply, however, that a solution will be reached by the Newton-Raphson method.

To establish the criteria for convergence of the Newton-Raphson method for all possible combinations of unknowns appears to be impractical, if not impossible. For the case of a network in which the elements are only pipes and valves, and for which the unknowns are the heads at joints, Warga[10] has shown that the system of equations has a unique solution, and that this solution is reached by the Newton-Raphson method provided that a reasonable starting assumption is made. The conditions to which the characteristic functions of the elements must conform in order to fit Warga's analysis are not fulfilled when the unknowns include pipe resistances, or when a pump is included in the network; a solution is, therefore, not guaranteed. This theoretical difficulty is overcome in practice by starting the solution with a good initial guess. For example, one often has a fairly good knowledge of pipe resistances, and if these are used as an initial guess, a solution is usually obtained.

If it is necessary to solve a network for a number of slightly different conditions, the amount of computation required may be greatly reduced by using the solution of one condition as the initial guess for a proceeding problem.

It is useful to understand some of the difficulties which may arise in the use of the Newton-Raphson method. Observe in Fig. 1 that, for the one-dimensional case, if x' happens to be the present value of x, the procedure would fail to yield a finite improvement Δx because the tangent of $f(x)$ at x' is parallel to the x-axis. Such singular points may exist in the multi-dimensional case as well, and would result in a matrix of derivatives in Eq. 16 with a rank smaller than NJ. (The computer program will print out a message to indicate that this condition has occurred.) Another condition which may arise is an oscillating correction at one or more nodes. Here, the correction computed at the $(k+1)^{st}$ iteration makes $x_{k+2} = x_k$—in other words, the path indicated in Fig. 1 is a closed loop. This can be corrected by dividing the computed correction by 2. A check for oscillating unknowns is included in the computer programs, and each oscillating correction is divided by 2.

As the solvability of the network and the convergence of the Newton-Raphson method from an arbitrary starting point cannot be proved, one has to anticipate the possibility of getting a message from the computer program that a solution can not be reached. As described, this message results from a singular matrix of derivatives in Eq. 16. When the message is received, a check is made to determine whether there was a node for which the continuity equation includes no unknowns. If this was not the case, a check of guesses for starting values of the unknowns is then made. As outlined above, for theoretical reasons, it would be advisable initially to suspect assumed pipe resistances and heads near pumps as the cause of trouble. After developing some experience with the physical network, subsequent trouble can be readily avoided. Invariably, a good initial guess will lead to the solution.

COMPARISON WITH THE HARDY CROSS METHOD

The Hardy Cross method is most commonly used for hand computations, although many computer programs exist for its execution by digital computers. The physical law governing the solution of the network is either continuity at all nodes (as used by the method presented above and by the Hardy Cross method of balancing heads) or continuity of the head line around each loop (as used by the Hardy Cross method of balancing flows). The equations for all nodes (or loops) must be satisfied simultaneously for the network to be "balanced", or solved.

Both the Hardy Cross and the present method solve these equations by iterations. The Hardy Cross performs iterations on separate equations, one at a time, while the Newton-Raphson method iterates on the set of equations simultaneously. The Hardy Cross method was developed to facilitate hand computations, and has the advantage of simplicity. The equation for each node (or loop) is formed and solved separately, independent of the other equations, and has the advantage that the amount of information necessary to form each equation is small, as it relates only to a small section of the network. The simplicity of the method is of help in programming the method, but much more important is the small amount of storage required by it. Because each equation is formulated using data for only one node (or loop), secondary storage (e.g., disk) can be used to advantage with only moderate increase in the time required to obtain a solution.

The Hardy Cross method suffers from a problem of solvability and convergence (Dillingham[7]). Various conditions, such as large pipe or very low flows, cause the iterative scheme to converge very slowly, or even diverge. Ad hoc procedures were developed (e.g., Dillingham[7]) to improve the convergence under some of these conditions, but there is no guarantee of convergence.

The general method of analysis developed herein requires the formulation and solution of the entire set of network equations. The large active computer space requirement for these computations makes the maximum size of network which can be handled by this method on any given computer smaller than what can be handled by the Hardy Cross method.

The advantages gained by the new method are great enough to outweigh this limitation for nearly all practical problems. The new method makes possible a direct solution for consumptions and resistances. It also makes the inclusion of elements other than pipes in the analysis a relatively simple and straightforward matter. To compute a set of unknown heads at joints, given all other information, is a simple matter with a computer program using the Hardy Cross method; but with the same program, to adjust the assumed resistances to make the computed pressure map correspond to measured field data is a tedious trial-and-error procedure.

Some computer programs using the Hardy Cross method require the balancing of flows initially before starting the iterations. If many solutions of a large network are required, this procedure can be excessively time-consuming. The flexibility of the Newton-Raphson method imposes no such restriction as long as the initial guesses are within reason. In a real-time environment, changes to fixed consumptions, heads, resistances, or locations and types of unknowns can be made rapidly and conveniently. In network design applications, the desirability of such flexibility is obvious. Moreover, for network monitoring or control, an area of increasing interest, flexibility is essential to compensate for temporary field sensor failures. When evaluating the two methods of analysis, one has to weigh their relative capabilities against the price paid for achieving them.

SENSITIVITY ANALYSIS

The study of a water distribution system includes investigating the effect of changes in heads, flows, and pipe resistances on the behavior of the network. When many such variables are changed simultaneously, such as the changes in consumption at all nodes from one time of the day to another, a complete new solution of the network has to be obtained.

There are, however, situations in which one is interested in the sensitivity of the network to changes in a single variable. This situation arises when adjusting network data to make the solution conform to field measurements, and in evaluating proposed modifications or planning operating rules.

If a network solution is available, a sensitivity analysis may be made without additional network solutions. The sensitivity analysis yields the rate of change of *NJ* selected variables with respect to changes in a single variable of interest. The same results may be used to evaluate the sensitivity of the variable of interest to changes in *NJ* other variables.

In a network with *NJ* nodes and *NL* pipes there are a total of $(2NJ + NL)$ variables (*NJ* heads, *NJ* consumptions, and *NL* resistances). The balanced network satisfies

$$F_j = 0 \qquad j = 1, \ldots, NJ \qquad (25)$$

We are interested in derivatives of the form $\partial x_i/\partial y_k$, in which x_i and y_k can each be a head, consumption, or pipe resistance. For the variable to be changed, y_k, we select a set, X, of *NJ* "free" variables which are allowed to change (X does not include y_k). The variables included in this set are denoted by x_i. Taking the derivative of Eq. 25 with respect to y_k,

$$\frac{dF_j}{dy_k} = 0 \qquad j = 1, \ldots, NJ \qquad (26)$$

Expansion of Eq. 26 yields

$$\frac{\partial F_j}{\partial y_k} + \sum_{x_i \in X} \frac{\partial F_j}{\partial x_i}\,\frac{\partial x_i}{\partial y_k} = 0 \qquad j = 1, \ldots, NJ \qquad (27)$$

which may be written in matrix form

$$\left[\frac{\partial F_j}{\partial x_i}\right]\left\{\frac{\partial x_i}{\partial y_k}\right\} = \left\{-\frac{\partial F_j}{\partial y_k}\right\} \qquad (28)$$

in which [] denotes a matrix and {} a column vector. The set of NJ equations given by Eq. 28 includes NJ unknowns—the partial derivatives $\partial x_i/\partial y_k$. The partial derivatives $\partial F_j/\partial y_k$ and $\partial F_j/\partial x_i$ are evaluated using data for the balanced network. The selection of the set of "free" variables, X, is subject to the same restrictions as the selection of unknowns for the network analysis.

The sensitivity analysis yields the approximate variations of x_i caused by a unit change in y_k, while all other $x \in X$ are allowed to change. The values are approximate, as they are derivatives at a point, rather than finite changes. They provide, however, a convenient and rapid way of comparing the effects of all possible changes.

The two examples below illustrate the use of the sensitivity analysis using the sample network of Fig. 2. Consider first the effect of a change in the resistance of pipe 5 on C_1 and H_2 through H_5. Thus $y_k = R_5 = R_{35}$ and $X = (C_1, H_2, H_3, H_4, H_5)$, and Eq. 28 takes on the form

$$\begin{bmatrix} \frac{\partial F_1}{\partial C_1} & \frac{\partial F_1}{\partial H_2} & 0 & 0 & \frac{\partial F_1}{\partial H_5} \\ 0 & \frac{\partial F_2}{\partial H_2} & \frac{\partial F_2}{\partial H_3} & \frac{\partial F_2}{\partial H_4} & 0 \\ 0 & \frac{\partial F_3}{\partial H_2} & \frac{\partial F_3}{\partial H_3} & \frac{\partial F_3}{\partial H_4} & \frac{\partial F_3}{\partial H_5} \\ 0 & \frac{\partial F_4}{\partial H_2} & \frac{\partial F_4}{\partial H_3} & \frac{\partial F_4}{\partial H_4} & \frac{\partial F_4}{\partial H_5} \\ 0 & 0 & \frac{\partial F_5}{\partial H_3} & \frac{\partial F_5}{\partial H_4} & \frac{\partial F_5}{\partial H_5} \end{bmatrix} \begin{bmatrix} \frac{\partial C_1}{\partial R_5} \\ \frac{\partial H_2}{\partial R_5} \\ \frac{\partial H_3}{\partial R_5} \\ \frac{\partial H_4}{\partial R_5} \\ \frac{\partial H_5}{\partial R_5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\partial F_3}{\partial R_5} \\ 0 \\ -\frac{\partial F_5}{\partial R_5} \end{bmatrix} \quad \ldots\ldots\ldots (29)$$

The expressions for the derivatives are given in Eqs. 13, 14, and 15. Eq. 29 is solved directly for the derivatives of all heads with respect to R_5. These derivatives give the change in heads due to a unit increase in R_5. A negative value means the head will decrease when R_5 is increased. Implicit in this solution is the assumption that H_1, C_2 through C_5, and all pipe resistances (except R_5, of course) remain unchanged when R_5 is changed.

In the next example, consider the problem of trying to increase the head at node 3 (say for fire demand). We select $X = (R_2, R_3, R_5, C_2, C_4)$, assuming that these variables may be controlled independently (for example by closing valves in the pipes). For $y_k = H_3$ Eq. 28 takes the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{\partial F_2}{\partial R_2} & 0 & 0 & \frac{\partial F_2}{\partial C_2} & 0 \\ \frac{\partial F_3}{\partial R_2} & \frac{\partial F_3}{\partial R_3} & \frac{\partial F_3}{\partial R_5} & 0 & 0 \\ 0 & \frac{\partial F_4}{\partial R_3} & 0 & 0 & \frac{\partial F_4}{\partial C_4} \\ 0 & 0 & \frac{\partial F_5}{\partial R_5} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial R_2}{\partial H_3} \\ \frac{\partial R_3}{\partial H_3} \\ \frac{\partial R_5}{\partial H_3} \\ \frac{\partial C_2}{\partial H_3} \\ \frac{\partial C_4}{\partial H_3} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\partial F_2}{\partial H_3} \\ -\frac{\partial F_3}{\partial H_3} \\ -\frac{\partial F_4}{\partial H_3} \\ -\frac{\partial F_5}{\partial H_3} \end{bmatrix} \quad \ldots\ldots\ldots (30)$$

which is solved for $\partial x_i/\partial H_3$, $x_i \in X$. As we are interested in increasing H_3, we examine the derivatives $\partial H_3/\partial x_i$, given by the reciprocals of the solutions of Eq. 30. A large positive $\partial H_3/\partial R_i$ means that closing a valve in pipe i (increasing R_i) will be effective in increasing H_3. A large (in absolute magnitude) negative $\partial H_3/\partial C_i$ means that decreasing C_i will be effective in increasing H_3.

IMPLEMENTATION ON A COMPUTER

Implementation of the method proceeded parallel with development. The set of computer programs written for this study included input-output programs, data organization programs, and analysis programs.

The input-output programs were required to minimize manual data handling. These programs were initially written for batch processing. When commercial time-sharing became available, the system programs, which were provided as part of the time-sharing service, eliminated the need for most of the special input-output programs. The data files were originally structured according to the uses which were forseen. As the study progressed and new uses for the data arose, it was necessary to develop data organization programs. These programs were used to temporarily restructure the data for specific uses. Analysis programs were written to perform correlation and multiple regression analysis on the field data, to generate synthetic data for filling incomplete records, to solve the steady-state network problem, and to investigate the sensitivity of the network.

All programs were implemented on large, high-speed computers (e.g., IBM 7090 for batch processing or DEC PDP6 for time-sharing). Except for the network analysis program (SDP), all other programs required small core space (less than 10K) and running time on the order of one minute.

The SDP program required approximately 15K for a network of 70 nodes and 100 pipes. Running time per iteration is roughly proportional to $(NJ)^2$. For a network with NJ = 35 and NL = 53, running time per iteration was approximately 6 sec on a time-shared DEC PDP6. With NJ = 55 and NL = 80 running time per iteration was approximately 15 sec on the same computer. These times include time spent on swapping in and out of core.

The number of iterations required to reach a solution, with a given allowable error, depends on the initial values of the unknowns. A maximum error of 0.10 cfs unbalanced discharge at any node was used throughout this study. With a good initial guess, the solution is usually reached in 10 to 15 iterations. In a series of related runs, as in a 24-hr simulated network operation, one or two iterations were often sufficient to achieve convergence after the first run (each run used the solution of the previous run as a starting point).

The implementation of a general steady-state network analysis program involves the development of a considerable number of smaller programs. These programs are major engineering assets for the study of water distribution systems. They provide the engineer with powerful tools for managing extremely large sets of data and for testing a comprehensive variety of network configurations and loadings.

CONCLUSIONS

The following conclusions may be made:

1. A generalized method for solving a steady-state nonlinear network has been developed.
2. The method requires a high-speed computer of large capacity for effective practical utilization. Such computers are now (1968) available commercially for on-line, time-shared use or for batch processing.
3. The method's flexibility for handling changes in input data can be best used in a time-shared environment as an on-line tool to assist engineering design and decision making.
4. The method as implemented allows selection of unknown heads, consumptions or pipe resistances.
5. Problems without physical solutions may be specified by an inexperienced user because of the wide range of unknown types and locations which the method will handle. In order to have some assurance that a solvable network is specified, it is convenient to specify that at least one of the following is unknown at each node: (a) The consumption at the node; (b) the head at the node or at an adjacent node; or (c) The resistance of a pipe connected to the node. In practical use this rule is not usually inconvenient.
6. The method converges to a solution rapidly if a good set of initial guesses is made for the values of the unknowns. If a set of solutions is desired for slightly different conditions, it is advantageous to arrange the runs in a sequence which minimizes radical changes in system performance from run to run.
7. In some applications, it may be desirable to incorporate a sensitivity analysis into the generalized method to avoid needless iteration to study slight variations from a particular solution. The sensitivity analysis appears to be of particular value for use in real-time network operation by computer.

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APPENDIX.--NOTATION

The following symbols are used in this paper:

C_j = consumption at node j, in cubic feet per second;
F_j = sum of the flows into node j in cubic feet per second;
H_j = head at node j, in feet;
NJ = number of nodes in the network;
NL = number of links (elements) in the network;
Q_{ji} = discharge from node i to node j, in cubic feet per second;
R_{ij} = resistance of the element connecting node i with node j; and
R_k = resistance of element number k.